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On Double Controlled Metric-Like Spaces and Related Fixed Point Theorems

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Abstract

In this paper, we generalize the fixed point theorem given in Mlaiki et al [Journal of Inequalities and Applications (2020) 2020:63] using the concept of double controlled metric-like spaces. Some examples are given here to illustrate the usability of the obtained results.

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In this article, we begin with the interesting generalization of the standard metric, so-called, b-metric. Bakhtin [1] and Czerwik [2] introduced b-metric spaces as a generalization of metric spaces and proved the contraction principle in this framework. Subsequently, many authors obtained fixed point results for single-valued or set-valued functions, in the setting of b-metric spaces. For detail see ([11],[12] , [13], [14], [15], [16], [19]). A good review on this topic is given by E. Karapinar [10].
In the extended b-metric definition of [3], a function $\theta : X \times X \to [1, \infty)$ is imposed instead of the constant $s \geq 1$. Motivated with this idea Mlaiki et al [4] introduced the controlled metric - type spaces (CMTS).

Following this, the natural question would come to mind. Could we choose two control functions here and get similar fixed point results? Answer is positive. Abdeljawad et al [6] defines the new metric so called double controlled metric-type spaces (DCMTS) in[2018].

Finally Mlaiki [5] choose two controlled functions and introduce double controlled metric-like spaces (DCMLS) in [2020] and then obtain some fixed point theorems by using Kannan contraction [17].

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In this article, we will focus to DCMLS's and get fixed point results on this space with Reich contraction.

1. Preliminaries

We now recollect some fundamental definitions, notations, and basic results that will be used throughout this paper.

Definition 1.1. *[1], [2] Let X be a non-empty set and $s \geq 1$. $d : X \times X \longrightarrow [0, \infty)$ be a mapping such that for all $x, y, z \in X$.*
(i) $x = y \Longrightarrow d(x, y) = 0$ (self- distance)
(ii) $d(x, y) = 0 \Longrightarrow x = y$ (indistancy)
(iii) $d(x, y) = d(y, x)$ (symmetric)
(iv) $d(x, y) \leq s\ [d(x, z) + d(z, y)]$ (weakened triangle inequality)

Then (X, d) is called a **b-metric** space (b-MS).

Definition 1.2. *[3] Let X be a non-empty set and $\theta : X \times X \to [1, \infty)$ be a function. A function $d : X \times X \longrightarrow [0, \infty)$ is called an **extended b-metric** if the follwing conditions are satisfied.*
(eb-i) $d(x, y) = 0 \Longleftrightarrow x = y$
(eb-ii) $d(x, y) = d(y, x)$
(eb-iii) $d(x, y) \leq \theta(x, y)\ [d(x, z) + d(z, y)]$
for all $x, y, z \in X$.

For $s = 1$ every b-metric satisfies the conditions of metric. But converse is not true.
Recently, some authors generalized the b-metric space to more general type of metric type spaces by using control functions in the triangle inequality.

Definition 1.3. *[4] Let X be a non-empty set $\theta : X \times X \to [1, \infty)$ be a function. A function $d : X \times X \longrightarrow [0, \infty)$ is called a **controlled metric type (CMT)** if the following conditions are satisfied.*
(cb-i) $d(x, y) = 0 \Longleftrightarrow x = y$
(cb-ii) $d(x, y) = d(y, x)$
(cb-iii) $d(x, y) \leq \theta(x, z)\ d(x, z) + \theta(z, y)\ d(z, y)]$
for all $x, y, z \in X$.

Abdeljawad et al then introduce a more general b-metric space, which is (DCMTS).

Definition 1.4. *[6] Let X be a non-empty set $\theta, \mu : X \times X \to [1, \infty)$ be a function. A function $d : X \times X \longrightarrow [0, \infty)$ is called a **double controlled metric type (DCMTS)** if the following conditions are satisfied.*
(cb-i) $d(x, y) = 0 \Longleftrightarrow x = y$
(cb-ii) $d(x, y) = d(y, x)$ (symmetric)
(cb-iii) $d(x, y) \leq \theta(x, z)\ d(x, z) + \mu(z, y)\ d(z, y)]$
for all $x, y, z \in X$.

Remark 1.1. *A controlled metric type is also a double controlled metric type when taking the $\theta = \mu$. The converse is not true in general.*

Finally, Mlaiki [5] present the generalization so called bouble controlled metric- like spaces (DCMLS).

Definition 1.5. *[5] Let X be a non-empty set $\theta, \mu : X \times X \to [1, \infty)$ be a function. A function $d : X \times X \longrightarrow [0, \infty)$ is called a **double controlled metric-like (DCMLS)** if the following conditions are satisfied.*
(db-i) $d(x, y) = 0 \Longrightarrow x = y$ (indistancy)
(db-ii) $d(x, y) = d(y, x)$
(db-iii) $d(x, y) \leq \theta(x, z)\ d(x, z) + \mu(z, y)\ d(z, y)$
for all $x, y, z \in X$.

Remark 1.2. *Any double controlled metric-type space (DCMTS) is a double controlled metric-like space (DCMLS). However, the converse is not true in general.*

2. Main Results

Theorem 2.1. *Let (X, d) be a complete double controlled metric-like space (DCMLS) with $\theta, \mu : X^2 \to [1, \infty)$ and T be a self mapping satisfying Reich condition. That is, T satisfies*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) \tag{1}$$

where $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$. Let $r = \frac{\alpha+\beta}{1-\gamma} < 1$ for all $x, y \in X$. For $x_0 \in X$, choose $x_n = T^n x_0$. Assume that

$$i) \sup_{m\geq 1} \lim_{i\to\infty} \frac{\theta(x_{i+1}, x_{i+2})}{\theta(x_i, x_{i+1})}.\mu(x_{i+1}, x_m) < \frac{1}{r} \tag{2}$$

$$ii) \lim_{n\to\infty} \theta(x, x_n) < \infty \quad \text{exist and finite and} \quad \lim_{n\to\infty} \mu(x, x_n) < \frac{1}{\gamma} \tag{3}$$

then T has a unique fixed point.

Proof. Let $x_0 \in X$. Consider the sequence $\{x_n\}$ with $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. It is clear that, if there exists n_0 for which $x_{n_0+1} = x_{n_0}$ then $Tx_{n_0} = x_{n_0}$. Then the proof is finished. Thus suppose that $x_{n+1} \neq x_n$ for every $n \in \mathbb{N}$. Therefore, we may assume that $x_{n+1} = x_n$ for all $n \in \mathbb{N}$. Now

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \leq \alpha\ d(x_{n-1}, x_n) + \beta\ d(x_{n-1}, Tx_{n-1}) \\ &+ \gamma\ d(x_n, Tx_n) \\ &= \alpha\ d(x_{n-1}, x_n) + \beta\ d(x_{n-1}, x_n) + \gamma\ d(x_n, x_{n+1}). \end{aligned} \tag{4}$$

Therefore , we get

$$d(x_n, x_{n+1}) \leq (\frac{\alpha+\beta}{1-\gamma})\ d(x_{n-1}, x_n) = r\ d(x_{n-1}, x_n). \tag{5}$$

Thus, we obtain

$$d(x_n, x_{n+1}) \leq r\ d(x_{n-1}, x_n) \leq r^2\ d(x_{n-2}, x_{n-1}) \leq \ldots \leq r^n\ d(x_0, x_1). \tag{6}$$

For all $n, m \in \mathbb{N}$ with $n < m$

$$\begin{aligned} d(x_n, x_m) &\leq \theta(x_n, x_{n+1})d(x_n, x_{n+1}) + \mu(x_{n+1}, x_m)d(x_{n+1}, x_m) \\ &\leq \theta(x_n, x_{n+1})d(x_n, x_{n+1}) + \mu(x_{n+1}, x_m)\theta(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) \\ &+ \mu(x_{n+1}, x_m)\mu(x_{n+2}, x_m)d(x_{n+2}, x_m) \\ &\leq \theta(x_n, x_{n+1})d(x_n, x_{n+1}) + \mu(x_{n+1}, x_m)\theta(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) \\ &+ \mu(x_{n+1}, x_m)\mu(x_{n+2}, x_m)\theta(x_{n+2}, x_{n+3})d(x_{n+2}, x_{n+3}) \\ &+ \mu(x_{n+1}, x_m)\mu(x_{n+2}, x_m)\theta(x_{n+3}, x_m)d(x_{n+3}, x_m) \leq \ldots \\ &\leq \theta(x_n, x_{n+1})d(x_n, x_{n+1}) + \sum_{i=n+1}^{m-2} (\prod_{j=n+1}^{i} \mu(x_j, x_m))\theta(x_i, x_{i+1})d(x_i, x_{i+1}) \\ &+ \prod_{i=n+1}^{m-1} \mu(x_i, x_m)d(x_{m-1}, x_m). \end{aligned} \tag{7}$$

Therefore, using (6) we get

$$\begin{aligned} d(x_n, x_m) \leq& \theta(x_n, x_{n+1})\ r^n d(x_0, x_1) \\ &+ \sum_{i=n+1}^{m-2} (\prod_{j=n+1}^{i} \mu(x_j, x_m))\theta(x_i, x_{i+1})\ r^i d(x_0, x_1) \\ &+ \prod_{i=n+1}^{m-1} \mu(x_i, x_m)\ r^{m-1} d(x_1, x_0) \end{aligned} \tag{8}$$

and then

$$\leq \theta(x_n, x_{n+1}) r^n d(x_0, x_1) + \sum_{i=n+1}^{m-1} (\prod_{j=0}^{i} \mu(x_j, x_m))\theta(x_i, x_{i+1})\ r^i d(x_0, x_1) \tag{9}$$

Now if we define

$$S_n = \sum_{i=0}^{n} (\prod_{j=0}^{i} \mu(x_j, x_m))\theta(x_i, x_{i+1})\ r^i\ d(x_0, x_1) \tag{10}$$

then applying the ratio test, we have

$$\begin{aligned} a_n = (\prod_{j=0}^{n} \mu(x_j, x_m))\theta(x_i, x_{i+1})\ r^i\ d(x_0, x_1) \\ \frac{a_{n+1}}{a_n} = r\ \mu(x_{i+1}, x_m)\ \frac{\theta(x_{i+1}, x_{i+2})}{\theta(x_i, x_{i+1})}. \end{aligned} \tag{11}$$

Therefore under condition (2), the series $\sum_n a_n$ converges. Therefore, $\lim_{n\to\infty} S_n$ exists. So the real sequence $\{S_n\}$ is Cauchy.

Thus we obtained the inequality

$$d(x_n, x_m) \leq d(x_1, x_0)[r^n \theta(x_n, x_{n+1}) + (S_{m-1} - S_n)] \tag{12}$$

Letting $n, m \to \infty$, we get

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0. \tag{13}$$

So, the sequence $\{x_n\}$ is d-Cauchy. Since (X, d) is a complete DCMLS then there is some $x_0^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x_0^*) = 0 \tag{14}$$

which means $x_n \to x_0^*$ as $n \to \infty$.

Now our claim is to show that $Tx_0^* = x_0^*$.

$$\begin{aligned} d(x_0^*, Tx_0^*) &\leq \theta(x_0^*, x_{n+1}) d(x_0^*, x_{n+1}) + \mu(x_{n+1}, Tx_0^*) d(x_{n+1}, Tx_0^*) \\ &= \theta(x_0^*, x_{n+1}) d(x_0^*, x_{n+1}) + \mu(x_{n+1}, Tx_0^*) d(T_n, Tx_0^*) \\ &\leq \theta(x_0^*, x_{n+1}) d(x_0^*, x_{n+1}) \\ &+ \mu(x_{n+1}, Tx_0^*)[\alpha d(x_n, Tx_0^*) + \beta d(x_n, Tx_n) + \gamma d(x_0^*, Tx_0^*) \\ &= \theta(x_0^*, x_{n+1}) d(x_0^*, x_{n+1}) \\ &+ \mu(x_{n+1}, Tx_0^*)[\alpha d(x_n, Tx_0^*) + \beta d(x_n, x_{n+1}) + \gamma d(x_0^*, Tx_0^*). \end{aligned} \tag{15}$$

Using the facts (ii) in (3) and letting the limit as $n \to \infty$ we obtained

$$d(x_0^*, Tx_0^*) \leq \mu(x_{n+1}, Tx_0^*)\ [\gamma \lim_{n\to\infty} d(x_0^*, Tx_0^*)]. \tag{16}$$

Suppose that $Tx_0^* \neq x_0^*$. Since $\lim_{n\to\infty} \mu(x, x_n) < \frac{1}{\gamma}$ we have

$$\begin{aligned} 0 < d(x_0^*, Tx_0^*) &\leq \mu(x_{n+1}, Tx_0^*)\ [\gamma d(x_0^*, Tx_0^*)] \\ &< d(x_0^*, Tx_0^*). \end{aligned} \tag{17}$$

It is a contradiction. Which means $x_0^* = Tx_0^*$.
Finally, assume that T has two fixed points, say p and q. Then

$$d(p, q) = d(Tp, Tq) \leq \alpha d(p, q) + \beta d(p, Tp) + \gamma d(q, Tq) \tag{18}$$

and so $d(p, q)(1-\alpha) \leq 0$. Since $\alpha \neq 1$ we received $d(p, q) = 0$ which implies $p = q$. This completes the proof.

Remark 2.1. *1. Our result is general then Mlaiki et al [4], 2018 and Abdeljawad et al [6], 2018. Their spaces satisfies both the conditions indistancy and self-distance.*
2. In the Reich contraction [9];

- *if we choose $\alpha = 0$, $\beta = 0$ then we obtain Banach contraction.*
- *If we choose $\alpha = \beta$ and $\gamma = 0$ then we obtain the Kannan contraction.*

3. Our result is general then the very recent work of J. Ahmad et al [18], 2020. Their Example 9 is not applicable to our theorem. They have only one controlled function and their space again satisfies both the conditions indistancy and self-distance.

4. Every partial metric space is a metric like space. Many papers can be cited in partial metric spaces. We'll refer [7] and [8].

Example 2.1. *In general, Reich contraction theorem is stronger than Banach's and Kannan's fixed point theorems.*
Let $X = [0, 1]$ be with usual metric and $T : [0, 1] \to [0, 1]$ be a mapping defined by

$$f(x) = \begin{cases} \frac{x}{3} & 0 \leq x < 1 \\ \frac{1}{6} & x = 1 \end{cases}$$

T does not satisfy Banach's condition, because it is not continuous at 1. Kannan's condition also cannot be satisfied because

$$d(T0, T\frac{x}{3}) = \frac{1}{2}[d(0, T0) + d(\frac{1}{3}, T\frac{1}{3})].$$

But it satisfies Reich contraction condition if we put $\alpha = \frac{1}{6},\ \beta = \frac{1}{9},\ \gamma = \frac{1}{3}$.

$\square$

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