http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 2, Pages 581–600 (2022)
DOI:10.31801/cfsuasmas.1027662
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received:November 23, 2021; Accepted: January 20, 2022

ORLICZ-LACUNARY CONVERGENT TRIPLE SEQUENCES AND IDEAL CONVERGENCE

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Abstract. In the present paper we introduce and study Orlicz lacunary convergent triple sequences over n-normed spaces. We make an effort to present the notion of g_3-ideal convergence in triple sequence spaces. We examine some topological and algebraic features of new formed sequence spaces. Some inclusion relations are obtained in this paper. Finally, we investigate ideal convergence in these spaces.

1. Introduction and Preliminaries

Several authors involving Duran [16], King [32], Lorentz [38], Moricz and Rhoades [46], Schafer [57] have worked the space of almost convergent sequences. The notion of strong almost convergence was considered by Maddox [39]. In [40], Maddox defined a generalization of strong almost convergence. Related articles with the topic almost convergence and strong almost convergence can be seen in [3, 8–15, 53].

In 1922, Banach defined normed linear spaces as a set of axioms. Since then, mathematicians keep on trying to find a proper generalization of this concept. The first notable attempt was by Vulich [60]. He introduced K-normed space in 1937. In another process of generalization, Siegfried Gähler [20] introduced 2-metric in 1963. As a continuation of his research, Gähler [21] proposed a mathematical structure, called 2-normed space, as a generalization of normed linear space which has been subsequently worked by many researchers [22–26, 33, 35, 41, 45].

In order to extend convergence of sequences, the notion of statistical convergence was given by Fast [17] for the real sequences. Afterward, it was further researched

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2020 *Mathematics Subject Classification.* Primary 40F05, 46A45; Secondary 40A05.

Keywords. Strongly almost convergence; almost convergence; n-norm; g_3-ideal convergence, lacunary sequence, Musielak-Orlicz function.

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from sequence point of view and connected with the summability theory (see [5, 7, 18, 31, 42–44, 47]) and has been generalized to the thought of 2-normed space by Gürdal and Pehlivan [27]. Recently, Alotaibi and Alroqi [1] extended this concept in paranormed space. The concept of paranorm is a generalization of absolute value (see [48]). We can refer to [4, 34] which are connected with this topic. The studies of double and triple sequences have seen rapid growth. The initial work on double sequences was establised by Bromwich [6]. The concept of regular convergence for double sequences was introduced by Hardy [29]. Quite recently, Zeltser [61] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Recently, Mursaleen and Edely [50] and Şahiner et al. [58] considered the idea of statistical convergence for multiple sequences. Fridy and Orhan [19] defined the concept of lacunary statistical convergence. The double lacunary statistical convergence was worked by Patterson and Savaş [55]. For details about definition of sequence spaces, Orlicz sequence spaces and paranormed spaces one can see [37, 51, 54, 56].

Since sequence convergence plays a very significant role in the essential theory of mathematics, there are many convergence notions in summability theory, in classical measure theory, in approximation theory, and in probability theory, and the relationships between them are examined. The concerned reader may consult Gürdal et al. [26], and Hazarika et al. [30], the monographs [2] and [49] for the background on the sequence spaces and related topics. Inspired by this, in this chapter, a further investigation into the mathematical features of triple sequences will be made. Section 2 recalls some known definitions and theorems in summability theory. In Section 3, we study the concept of Musielak-Orlicz lacunary almost and strongly almost convergent triple sequences over n-normed spaces and introduce the notion of g_3-ideal convergence in a paranormed triple sequence spaces, where the base space is an n-normed space. In addition we investigate some topological and algebraic features of newly formed sequence spaces. In addition to these definitions, natural inclusion theorems shall also be presented.

Now we remind the n-normed space which was determined in [23] and some definitions on n-normed space (see [28]).

Definition 1. *Let $n \in \mathbb{N}$ and X be a real vector space of dimension $d \geq n$ (Here we allow d to be infinite). A real-valued function $\|.,...,.\|$ on X^n satisfying the subsequent four features*

(i) $\|x_1, x_2, ..., x_n\| = 0$ *iff $x_1, x_2, ..., x_n$ are linearly dependent;*

(ii) $\|x_1, x_2, ..., x_n\|$ *is invariant under permutation;*

(iii) $\|x_1, x_2, ..., x_{n-1}, \alpha x_n\| = |\alpha| \ \|x_1, x_2, ..., x_{n-1}, x_n\|$, *for any $\alpha \in \mathbb{R}$;*

(iv) $\|x_1, x_2, ..., x_{n-1}, y + z\| \leq \|x_1, x_2, ..., x_{n-1}, y\| + \|x_1, x_2, ..., x_{n-1}, z\|$,

is called an n-norm on X and the pair $(X, \|.,...,.\|)$ is called an n-normed space.

It is well-known fact from the following corollary that n-normed spaces are actually normed spaces.

Corollary 1. *([23]) Every n-normed space is an $(n-r)$-normed space for all $r = 1, ..., n-1$. Especially, every n-normed space is a normed space.*

Example 1. *A standard example of an n-normed space is $X = \mathbb{R}^n$ equipped with the n-norm is*

$\|x_1, x_2, ..., x_{n-1}, x_n\| :=$ *the volume of the n-dimensional parallelepiped spanned by $x_1, x_2, ..., x_{n-1}, x_n$ in X.*

Observe that in any n-normed space $(X, \|., ..., .\|)$ we acquire

$$\|x_1, x_2, ..., x_{n-1}, x_n\| \geq 0$$

and

$$\|x_1, x_2, ..., x_{n-1}, x_n\| = \|x_1, x_2, ..., x_{n-1}, x_n + \alpha_1 x_1 + ... + \alpha_{n-1} x_{n-1}\|$$

for all $x_1, x_2, ..., x_n \in X$ and $\alpha_1, ..., \alpha_{n-1} \in \mathbb{R}$.

Definition 2. *A sequence (x_k) in an n-normed space $(X, \|., ..., .\|)$ is said to convergent to an $l \in X$ if*

$$\lim_{k\to\infty} \|x_k - l, y_1, y_2, ..., y_{n-1}\| = 0$$

for every $y_1, y_2, ..., y_{n-1} \in X$.

Definition 3. *A sequence (x_k) in an n-normed space $(X, \|., ..., .\|)$ is called a Cauchy sequence if*

$$\lim_{k,l\to\infty} \|x_k - x_l, y_1, y_2, ..., y_{n-1}\| = 0$$

for every $y_1, y_2, ..., y_{n-1} \in X$.

By the convergence of a triple sequence we mean the convergence in the Pringsheim sense, i.e. a triple sequence $x = (x_{ijk})$ has Pringsheim limit ξ (indicated by $P - \lim x = \xi$) provided that given $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $|x_{ijk} - \xi| < \varepsilon$ wherever $i, j, k > n$, (see [58]). We shall denote more briefly as $P-$convergent. The triple sequence $x = (x_{ijk})$ is bounded if there is $K > 0$ such that $|x_{ijk}| < K$ for all i, j and k.

Definition 4. *A subset K of $\mathbb{N}^3$ is called to have natural density $\delta_3(K)$ if*

$$\delta_3(K) = P - \lim_{n,k,l\to\infty} \frac{|K_{nkl}|}{nkl}$$

exists, where the vertical bars signify the number of (n, k, l) in K such that $p \leq n$, $q \leq k$, $r \leq l$. Then, a real triple sequence $x = (x_{pqr})$ is named to be statistically convergent to ξ in Pringsheim's sense provided that for every $\varepsilon > 0$,

$$\delta_3\left(\left\{(n, k, l) \in \mathbb{N}^3 : p \leq n, q \leq k, r \leq l, \ |x_{pqr} - \xi| \geq \varepsilon\right\}\right) = 0.$$

Statistical convergence was further generalized in the paper [36] utilizing the notion of an ideal of subsets of the set $\mathbb{N}$. We say that a non-empty family of sets $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is an ideal on $\mathbb{N}$ if $\mathcal{I}$ is hereditary (i.e. $B \subset A \in \mathcal{I} \Rightarrow B \in \mathcal{I}$) and additive (i.e. $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$). An ideal $\mathcal{I}$ on $\mathbb{N}$ for which $\mathcal{I} \neq \mathcal{P}(\mathbb{N})$ is named a non-trivial ideal. A non-trivial ideal $\mathcal{I}$ is named admissible if $\mathcal{I}$ includes all finite subsets of $\mathbb{N}$. If not otherwise stated in the sequel $\mathcal{I}$ will signify an admissible ideal. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be any ideal. A class $\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \backslash A\}$ named the filter connected with the ideal $\mathcal{I}$, is a filter on $\mathbb{N}$.

Definition 5. *Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$ and $x = (x_k)$ be a real sequence. We say that the sequence x is $\mathcal{I}$-convergent to $\xi \in \mathbb{R}$ provided that for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in \mathbb{N} : |x_k - \xi| \geq \varepsilon\} \in \mathcal{I}$.*

Take for $\mathcal{I}$ the class $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is a non-trivial admissible ideal and $\mathcal{I}_f$-convergence coincides with the usual convergence. For more information about $\mathcal{I}$-convergent, see the references in [52].

Definition 6. *([59]) Let $\mathcal{I}_3$ be an admissible ideal on $\mathbb{N}^3$, then a triple sequence (x_{jkl}) is named to be $\mathcal{I}_3$-convergent to ξ in Pringsheim's sense if for every $\varepsilon > 0$,*

$$\left\{(j,k,l) \in \mathbb{N}^3 : |x_{jkl} - \xi| \geq \varepsilon\right\} \in \mathcal{I}_3.$$

In this case, one writes $\mathcal{I}_3\text{-}\lim x_{jkl} = \xi$.

Remark 1. *(i) Let $\mathcal{I}_3(f)$ be the family of all finite subsets of $\mathbb{N}^3$. Then $\mathcal{I}_3(f)$ convergence coincides with the convergence of triple sequences in [58].*

(ii) Let $\mathcal{I}_3(\delta) = \left\{A \subset \mathbb{N}^3 : \delta_3(A) = 0\right\}$. Then $\mathcal{I}_3(\delta)$ convergence coincides with the statistical convergence in [58].

2. Main Results

Following the above definitions and results, we aim in this section to introduce some new notions of Orlicz lacunary convergent triple sequences and g_3-ideal convergence over n-normed spaces. In addition to these definition, also some topological and algebraic properties of newly formed sequence spaces have been established.

A triple sequence $x = (x_{ijk})$ of real numbers is called to be almost convergent to a limit ξ if

$$\lim_{p,q,r \to \infty} \sup_{\alpha,\beta,\gamma \geq 0} \left| \frac{1}{pqr} \sum_{i=\alpha}^{\alpha+p-1} \sum_{j=\beta}^{\beta+q-1} \sum_{k=\gamma}^{\gamma+r-1} (x_{ijk} - \xi) \right| = 0. \tag{1}$$

In this case, ξ is called the f_3-limit of x and the space of all almost convergent triple sequences is denoted by f_3,

$$f_3 = \left\{ x = (x_{ijk}) : \lim_{p,q,r \to \infty} |h_{pqr\alpha\beta\gamma}(x) - \xi| = 0, \text{ uniformly in } \alpha, \beta, \gamma \right\},$$

where

$$h_{pqr\alpha\beta\gamma}(x) = \frac{1}{(p+1)(q+1)(r+1)} \sum_{i=0}^{p} \sum_{j=0}^{q} \sum_{k=0}^{r} x_{i+\alpha,j+\beta,k+\gamma}.$$

The set of all strongly almost convergent triple sequences is denoted by $[f_3]$.

Let $\mathcal{M} = (M_{mno})$ be a Musielak-Orlicz function, $u = (u_{mno})$ be a triple sequence of positive real numbers and $p = (p_{mno})$ be a bounded sequence of positive real numbers. We indicate the space of all sequences defined over $(X, \|., ..., .\|)$ by $w(n-X)$. Now we itendify the following sequence spaces for some ρ and for every nonzero $y_1, y_2, ..., y_{n-1} \in X$:

$$[M, u, F, p, \|., ..., .\|] = \{x \in w(n-X) :$$
$$\lim_{p,q,r\to\infty} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno}\left(\left\|\frac{h_{pqr\alpha\beta\gamma}(x-\xi)}{\rho}, y_1, ..., y_{n-1}\right\|\right)\right]^{p_{mno}} = 0,$$
$$\text{uniformly in } \alpha, \beta, \gamma \geq 1\}$$

and

$$[M, u, [F], p, \|., ..., .\|] = \{x \in w(n-X) :$$
$$\lim_{p,q,r\to\infty} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno}\left(h_{pqr\alpha\beta\gamma}\left(\left\|\frac{x-\xi}{\rho}, y_1, ..., y_{n-1}\right\|\right)\right)\right]^{p_{mno}} = 0,$$
$$\text{uniformly in } \alpha, \beta, \gamma \geq 1\},$$

where $h_{pqr\alpha\beta\gamma}(x)$ is defined as in (1). We write $[\mathcal{M}, u, [F], p, \|., ..., .\|] - \lim x = \xi$. Also we have

$$[\mathcal{M}, u, [F], p, \|., ..., .\|] \subset [\mathcal{M}, u, F, p, \|., ..., .\|] \subset [M, u, \ell^{\infty}, p, \|., ..., .\|]$$

holds from the inequality:

$$\left\|\frac{h_{pqr\alpha\beta\gamma}(x-\xi)}{\rho}, y_1, ..., y_{n-1}\right\| = \left\|\frac{\frac{1}{(p+1)(q+1)(r+1)} \sum_{i=0}^{p} \sum_{j=0}^{q} \sum_{k=0}^{r} (x_{i+\alpha,j+\beta,k+\gamma} - \xi)}{\rho}, y_1, ..., y_{n-1}\right\|$$
$$\leq \frac{1}{(p+1)(q+1)(r+1)} \sum_{i=0}^{p} \sum_{j=0}^{q} \sum_{k=0}^{r} \left\|\frac{x_{i+\alpha,j+\beta,k+\gamma} - \xi}{\rho}, y_1, ..., y_{n-1}\right\| = h_{pqr\alpha\beta\gamma}\left(\left\|\frac{x-\xi}{\rho}, y_1, ..., y_{n-1}\right\|\right).$$

Furthermore, the triple sequence $\theta_3 = \theta_{v,\eta,\tau} = \{(m_v, n_\eta, o_\tau)\}$ is called triple lacunary sequence if there exist three increasing sequences of integers such that

$$m_0 = 0, \; h_v = m_v - m_{v-1} \to \infty \text{ as } v \to \infty,$$

$$n_0 = 0, \; h_\eta = n_\eta - n_{\eta-1} \to \infty \text{ as } \eta \to \infty,$$

and

$$o_0 = 0, \; h_\tau = o_\tau - o_{\tau-1} \to \infty \text{ as } \tau \to \infty.$$

Let $m_{v,\eta,\tau} = m_v n_\eta o_\tau$, $h_{v,\eta,\tau} = h_v h_\eta h_\tau$ and $I_{v,\eta,\tau}$ is determined as follows:

$$I_{v,\eta,\tau} = \{(m, n, o) : m_{v-1} < m \leq m_v, n_{\eta-1} < n \leq n_\eta \text{ and } o_{\tau-1} < o \leq o_\tau\},$$

$$s_v = \frac{m_v}{m_{v-1}}, s_\eta = \frac{n_\eta}{n_{\eta-1}},\ s_r = \frac{o_\tau}{o_{\tau-1}} \text{ and } s_{v,\eta,\tau} = s_v s_\eta s_\tau.$$

Let $D \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$. The number

$$\delta_3^{\theta_3}(D) = \lim_{v,\eta,\tau} \frac{1}{h_{v,\eta,\tau}} \left|\{(m,n,o) \in I_{v,\eta,\tau} : (m,n,o) \in D\}\right|$$

is named to be the θ_3-density of D, provided the limit exists.

The spaces of lacunary almost and strongly almost convergent triple sequences in n-normed spaces are identified as follows:

$$\begin{aligned}&[M,u,F_\theta,p,\|.,...,.\|] = \{x \in (x_{ijk}) \in w(n-X) :\\ &\lim_{v,\eta,\tau\to\infty} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left\| \frac{1}{h_{v,\eta,\tau}} \sum_{i,j,k\in I_{v\eta\tau}} \left(\frac{x_{i+\alpha,j+\beta,k+\gamma}-\xi}{\rho}, y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}\\ &= 0, \text{ uniformly in } \alpha,\beta,\gamma \geq 1\}\end{aligned}$$

and

$$\begin{aligned}&[M,u,[F_\theta],p,\|.,...,.\|] = \{x \in (x_{ijk}) \in w(n-X) :\\ &\lim_{v,\eta,\tau\to\infty} \frac{1}{h_{v,\eta,\tau}} \sum_{i,j,k\in I_{v\eta\tau}} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left\|\left(\frac{x_{i+\alpha,j+\beta,k+\gamma}-\xi}{\rho}, y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}\\ &= 0, \text{ uniformly in } \alpha,\beta,\gamma \geq 1\},\end{aligned}$$

where

$$F_\theta = \left\{ \begin{array}{c} x = (x_{ijk}) : \lim_{v,\eta,\tau\to\infty} \left\| \frac{1}{h_{v,\eta,\tau}} \sum_{i,j,k\in I_{v\eta\tau}} (x_{i+\alpha,j+\beta,k+\gamma}-\xi), y_1,...,y_{n-1}\right\|, \\ \text{uniformly in } \alpha,\beta,\gamma \end{array} \right\}$$

and

$$[F_\theta] = \left\{ \begin{array}{c} x = (x_{ijk}) : \lim_{v,\eta,\tau\to\infty} \frac{1}{h_{v,\eta,\tau}} \sum_{i,j,k\in I_{v\eta\tau}} \|(x_{i+\alpha,j+\beta,k+\gamma}-\xi), y_1,...,y_{n-1}\|, \\ \text{uniformly in } \alpha,\beta,\gamma \end{array} \right\}.$$

Lemma 1. *For a given $\varepsilon > 0$ and let $x = (x_{ijk})$ be a strongly almost convergent triple sequence. Then there exist p_0, q_0, r_0, α_0, β_0 and γ_0 such that*

$$\frac{1}{pqr} \sum_{i=\alpha}^{\alpha+p-1} \sum_{j=\beta}^{\beta+q-1} \sum_{k=\gamma}^{\gamma+r-1} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\|\frac{x_{i,j,k}-\xi}{\rho}, y_1,...,y_{n-1}\right\|\right)\right]^{p_{mno}} < \varepsilon$$

for all $p_{mno} \geq 1$, $p \geq p_0$, $q \geq q_0$, $r \geq r_0$, $\alpha \geq \alpha_0$, $\beta \geq \beta_0$, $\gamma \geq \gamma_0$, for every nonzero $y_1,...,y_{n-1} \in X$. Then $x \in [\mathcal{M},u,[F],p,\|.,...,.\|]$.

Proof. Given $\varepsilon > 0$. Take p_0', q_0', r_0', α_0, β_0, γ_0 such that

$$\frac{1}{pqr} \sum_{i=\alpha}^{\alpha+p-1} \sum_{j=\beta}^{\beta+q-1} \sum_{k=\gamma}^{\gamma+r-1} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\|\frac{x_{i,j,k}-\xi}{\rho}, y_1,...,y_{n-1}\right\|\right)\right]^{p_{mno}} < \frac{\varepsilon}{2} \tag{2}$$

for all $p \geq p_0'$, $q \geq q_0'$, $r \geq r_0'$, $\alpha \geq \alpha_0$, $\beta \geq \beta_0$, $\gamma \geq \gamma_0$. Now we have to prove only that there are p_0'', q_0'', r_0'' such that for $p > p_0''$, $q > q_0''$, $r > r_0''$, $0 \leq \alpha \leq \alpha_0$, $0 \leq \beta \leq \beta_0$, $0 \leq \gamma \leq \gamma_0$.

$$\frac{1}{pqr} \sum_{i=\alpha}^{\alpha+p-1} \sum_{j=\beta}^{\beta+q-1} \sum_{k=\gamma}^{\gamma+r-1} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\| \frac{x_{i,j,k} - \xi}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right]^{p_{mno}} < \varepsilon. \tag{3}$$

By selecting $p_0 = \max\left(p_0', p_0''\right)$, $q_0 = \max\left(q_0', q_0''\right)$ and $r_0 = \max\left(r_0', r_0''\right)$, (3) will holds for $p \geq p_0$, $q \geq q_0$, $r \geq r_0$ and $\forall \alpha, \beta, \gamma$. Take α_0, β_0, γ_0 be fixed,

$$\sum_{i=0}^{\alpha_0-1} \sum_{j=0}^{\beta_0-1} \sum_{k=0}^{\gamma_0-1} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\| \frac{x_{i,j,k} - \xi}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right]^{p_{mno}} = K_1. \tag{4}$$

$$\sum_{i=\alpha}^{\alpha_0-1} \sum_{j=\beta}^{\beta_0-1} \sum_{k=\gamma_0}^{\gamma+r-1} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\| \frac{x_{i,j,k} - \xi}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right]^{p_{mno}} = K_2. \tag{5}$$

$$\sum_{i=\alpha}^{\alpha_0-1} \sum_{j=\beta_0}^{\beta+q-1} \sum_{k=\gamma}^{\gamma_0-1} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\| \frac{x_{i,j,k} - \xi}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right]^{p_{mno}} = K_3. \tag{6}$$

$$\sum_{i=\alpha}^{\alpha_0-1} \sum_{j=\beta_0}^{\beta+q-1} \sum_{k=\gamma_0}^{\gamma+r-1} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\| \frac{x_{i,j,k} - \xi}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right]^{p_{mno}} = K_4. \tag{7}$$

$$\sum_{i=\alpha_0}^{\alpha+p-1} \sum_{j=\beta}^{\beta_0-1} \sum_{k=\gamma}^{\gamma_0-1} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\| \frac{x_{i,j,k} - \xi}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right]^{p_{mno}} = K_5. \tag{8}$$

$$\sum_{i=\alpha_0}^{\alpha+p-1} \sum_{j=\beta}^{\beta_0-1} \sum_{k=\gamma_0}^{\gamma+r-1} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\| \frac{x_{i,j,k} - \xi}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right]^{p_{mno}} = K_6. \tag{9}$$

$$\sum_{i=\alpha_0}^{\alpha+p-1} \sum_{j=\beta_0}^{\beta+q-1} \sum_{k=\gamma}^{\gamma_0-1} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\| \frac{x_{i,j,k} - \xi}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right]^{p_{mno}} = K_7. \tag{10}$$

Now taking $0 \leq \alpha \leq \alpha_0$, $0 \leq \beta \leq \beta_0$, $0 \leq \gamma \leq \gamma_0$ and $p > \alpha_0$, $q > \beta_0$, $r > \gamma_0$, we have from (4-10)

$$\tfrac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left(\left\|\tfrac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right\|\right)\right]^{p_{mno}}$$

$$= \tfrac{1}{pqr}\left(\sum_{i=\alpha}^{\alpha_0-1}+\sum_{i=\alpha_0}^{\alpha+p-1}\right)\left(\sum_{j=\beta}^{\beta_0-1}+\sum_{j=\beta_0}^{\beta+q-1}\right)\left(\sum_{k=\gamma}^{\gamma_0-1}+\sum_{k=\gamma_0}^{\gamma+r-1}\right)\times$$

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left(\left\|\tfrac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right\|\right)\right]^{p_{mno}}$$

$$\leq \tfrac{K_1}{pqr}+\tfrac{K_2}{pqr}+\tfrac{K_3}{pqr}+\tfrac{K_4}{pqr}+\tfrac{K_5}{pqr}+\tfrac{K_6}{pqr}+\tfrac{K_7}{pqr}+$$

$$+\tfrac{1}{pqr}\sum_{i=\alpha_0}^{\alpha_0+p-1}\sum_{j=\beta_0}^{\beta_0+q-1}\sum_{k=\gamma_0}^{\gamma_0+r-1}\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left(\left\|\tfrac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right\|\right)\right]^{p_{mno}}$$

$$\leq \tfrac{K_1}{pqr}+\tfrac{K_2}{pqr}+\tfrac{K_3}{pqr}+\tfrac{K_4}{pqr}+\tfrac{K_5}{pqr}+\tfrac{K_6}{pqr}+\tfrac{K_7}{pqr}+\tfrac{\varepsilon}{2} \text{ from (2).}$$

Taking p, q, r sufficiently large, we can obtain

$$\frac{K_1}{pqr}+\frac{K_2}{pqr}+\frac{K_3}{pqr}+\frac{K_4}{pqr}+\frac{K_5}{pqr}+\frac{K_6}{pqr}+\frac{K_7}{pqr}+\frac{\varepsilon}{2}<\varepsilon.$$

gives (3). This concludes the proof. □

Theorem 1. *Let* $p_{mno} \geq 1\ \forall\ m, n, o$ *and for every* θ_3 *we have* $[\mathcal{M}, u, [F_\theta], p, \|., ..., .\|] = [\mathcal{M}, u, [F], p, \|., ..., .\|]$.

Proof. Let $\{x_{ijk}\} \in [\mathcal{M}, u, [F_\theta], p, \|., ..., .\|]$. Then for given $\varepsilon > 0$, there exist p_0, q_0, r_0 and ξ such that

$$\frac{1}{h_{\upsilon,\eta,\tau}}\sum_{i=\alpha}^{\alpha+h_\upsilon-1}\sum_{j=\beta}^{\beta+h_\eta-1}\sum_{k=\gamma}^{\gamma+h_\tau-1}\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left(\left\|\frac{x_{i,j,k}-L}{\rho},y_1,...,y_{n-1}\right\|\right)\right]^{p_{mno}}<\varepsilon$$

for $\upsilon \geq \upsilon_0$, $\eta \geq \eta_0$, $\tau \geq \tau_0$ and $\alpha = U_{\upsilon-1}+1+a$, $\beta = V_{\eta-1}+1+a$, $\gamma = Z_{\tau-1}+1+a$, $a \geq 0$. Let $p \geq h_\upsilon$, $q \geq h_\eta$, $r \geq h_\tau$ write $p = ch_\upsilon + \theta$, $q = bh_\eta + \theta$, $r = dh_\tau + \theta$

where b, c, d are integers. Since b, c, $d \geq 1$. Now

$$\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left(\left\|\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right\|\right)\right]^{p_{mno}}$$

$$\leq \frac{1}{pqr}\sum_{i=\alpha}^{\alpha+(c+1)h_\upsilon-1}\sum_{j=\beta}^{\beta+(b+1)h_\eta-1}\sum_{j=\gamma}^{\gamma+(d+1)h_\tau-1}\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left(\left\|\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right\|\right)\right]^{p_{mno}}$$

$$= \frac{1}{pqr}\sum_{u'=0}^{c}\sum_{i=\alpha+u'h_\upsilon}^{\alpha+\left(u'+1\right)h_\upsilon-1}\sum_{u'=0}^{b}\sum_{j=\beta+u'h_\eta}^{\beta+\left(u'+1\right)h_\eta-1}\sum_{u'=0}^{d}\sum_{j=\gamma+\upsilon'h_\tau}^{\gamma+\left(u'+1\right)h_\tau-1}\times$$

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left(\left\|\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right\|\right)\right]^{p_{mno}}$$

$$\leq \frac{(d+1)(c+1)(b+1)}{pqr}h_\upsilon h_\eta h_\tau\varepsilon \leq \frac{4cbdh_\upsilon h_\eta h_\tau\varepsilon}{pqr}\ \ (d,c,b\geq 1).$$

For $\frac{h_\upsilon h_\eta h_\tau}{pqr}\leq 1$, since $\frac{cbdh_\upsilon h_\eta h_\tau}{pqr}\leq 1$

$$\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left(\left\|\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right\|\right)\right]^{p_{mno}}\leq 4\varepsilon.$$

Using Lemma 1, we get $[\mathcal{M},u,[F_\theta],p,\|.,...,.\|]\subseteq[\mathcal{M},u,[F],p,\|.,...,.\|]$. □

Lemma 2. *Assume for a given $\varepsilon>0$ there exist p_0, q_0, r_0 and α_0, β_0, γ_0 such that*

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}<\varepsilon$$

for all $p\geq p_0$, $q\geq q_0$, $r\geq r_0$, $\alpha\geq\alpha_0$, $\beta\geq\beta_0$, $\gamma\geq\gamma_0$, for every nonzero $y_1,...,y_{n-1}\in X$ and for some $\rho>0$. Then $x\in[\mathcal{M},F,u,p,\|.,...,.\|]$.

Proof. Assume $\varepsilon>0$. Take p_0', q_0', r_0', α_0, β_0, γ_0 such that

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}<\frac{\varepsilon}{2} \tag{11}$$

for all $p\geq p_0'$, $q\geq q_0'$, $r\geq r_0'$, $\alpha\geq\alpha_0$, $\beta\geq\beta_0$, $\gamma\geq\gamma_0$. By Lemma 1, it is enought to denote that there exist p_0'', q_0'', r_0'' such that for $p\geq p_0''$, $q\geq q_0''$, $r\geq r_0''$, $0\leq\alpha\leq\alpha_0$, $0\leq\beta\leq\beta_0$, $0\leq\gamma\leq\gamma_0$

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}<\varepsilon$$

Since α_0, β_0, γ_0 are fixed, let $0 \leq \alpha \leq \alpha_0$, $0 \leq \beta \leq \beta_0$, $0 \leq \gamma \leq \gamma_0$ and $p > \alpha_0$, $q > \beta_0$, $r > \gamma_0$. According to (4-10), we obtain

$$\begin{aligned}
&\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}\\
\leq\ &\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha_0-1}\sum_{j=\beta}^{\beta_0-1}\sum_{k=\gamma}^{\gamma_0-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}\\
&+\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha_0-1}\sum_{j=\beta}^{\beta_0-1}\sum_{k=\gamma_0}^{\gamma+r-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}\\
&+\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha_0-1}\sum_{j=\beta_0}^{\beta+q-1}\sum_{k=\gamma}^{\gamma_0-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}\\
&+\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha_0-1}\sum_{j=\beta_0}^{\beta+q-1}\sum_{k=\gamma_0}^{\gamma+r-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}\\
&+\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha_0}^{\alpha+p-1}\sum_{j=\beta}^{\beta_0-1}\sum_{k=\gamma}^{\gamma_0-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}\\
&+\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha_0}^{\alpha+p-1}\sum_{j=\beta_0}^{\beta+q-1}\sum_{k=\gamma}^{\gamma_0-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}\\
&+\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha_0}^{\alpha+p-1}\sum_{j=\beta_0}^{\beta+q-1}\sum_{k=\gamma_0}^{\gamma+r-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}
\end{aligned}$$

$$\begin{aligned}
&\leq \tfrac{K_1}{pqr}+\tfrac{K_2}{pqr}+\tfrac{K_3}{pqr}+\tfrac{K_4}{pqr}+\tfrac{K_5}{pqr}+\tfrac{K_6}{pqr}+\tfrac{K_7}{pqr}+\\
&+\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\tfrac{1}{pqr}\sum_{i=\alpha_0}^{\alpha+p-1}\sum_{j=\beta_0}^{\beta+q-1}\sum_{k=\gamma_0}^{\gamma+r-1}\left(\tfrac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}
\end{aligned}\tag{12}$$

Let $p-\alpha_0 > p'_0$, $q-\beta_0 > q'_0$, $r-\gamma_0 > r'_0$. Then, for $0 \leq \alpha \leq \alpha_0$, $0 \leq \beta \leq \beta_0$, $0 \leq \gamma \leq \gamma_0$, we get $p+\alpha-\alpha_0 \geq p'_0$, $q+\beta-\beta_0 \geq q'_0$, $r+\gamma-\gamma_0 \geq r'_0$. From (11), we have

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{(p+\alpha+\alpha_0)(q+\beta+\beta_0)(r+\gamma+\gamma_0)}\sum_{i=\alpha_0}^{\alpha_0+p+\alpha-\alpha_0}\sum_{j=\beta_0}^{\beta_0+q+\beta-\beta_0}\sum_{k=\gamma_0}^{\gamma_0+r+\gamma-\gamma_0}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}<\frac{\varepsilon}{2}. \tag{13}$$

From (12) and (13), we have

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}$$

$$\leq \frac{K_1}{pqr}+\frac{K_2}{pqr}+\frac{K_3}{pqr}+\frac{K_4}{pqr}+\frac{K_5}{pqr}+\frac{K_6}{pqr}+\frac{K_7}{pqr}$$

$$+\frac{(p+\alpha-\alpha_0)(q+\beta-\beta_0)(r+\gamma-\gamma_0)}{pqr}\frac{\varepsilon}{2}$$

$$\leq \frac{K_1}{pqr}+\frac{K_2}{pqr}+\frac{K_3}{pqr}+\frac{K_4}{pqr}+\frac{K_5}{pqr}+\frac{K_6}{pqr}+\frac{K_7}{pqr}+\frac{\varepsilon}{2}<\varepsilon,$$

for sufficiently large p,q,r. □

Theorem 2. *(i) For every θ,*
$[\mathcal{M},u,F_\theta,p,\|.,...,.\|]\cap[\mathcal{M},u,l^\infty,p,\|.,...,.\|]=[\mathcal{M},u,F,p,\|.,...,.\|]$.
(ii) For every θ, $[\mathcal{M},u,F_\theta,p,\|.,...,.\|]\nsubseteq[\mathcal{M},u,l^\infty,p,\|.,...,.\|]$.

Proof. (i) Let $\{x_{ijk}\}\in[\mathcal{M},u,F_\theta,p,\|.,...,.\|]\cap[\mathcal{M},u,l^\infty,p,\|.,...,.\|]$ $\forall\varepsilon>0$ there exist υ_0, η_0 and τ_0 such that

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{h_{\upsilon,\eta,\tau}}\sum_{i=\alpha}^{\alpha+h_\upsilon-1}\sum_{j=\beta}^{\beta+h_\eta-1}\sum_{k=\gamma}^{\gamma+h_\tau-1}\left(\frac{x_{i,j,k}-L}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}<\frac{\varepsilon}{2} \tag{14}$$

for $\upsilon,\eta,\tau \geq \upsilon_0,\eta_0,\tau_0$, $\alpha \geq \alpha_0$, $\beta \geq \beta_0$, $\gamma \geq \gamma_0$, $\alpha = U_{\upsilon-1}+1+a$, $\beta = V_{\eta-1}+1+a$, $\gamma = Z_{\tau-1}+1+a$, $a \geq 0$. Let integers $p \geq h_\upsilon$, $q \geq h_\eta$, $r \geq h_\tau$, $b,c,d \geq 1$. Then

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}M_{mno}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}$$

$$\leq \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{\mu=0}^{c-1}\sum_{i=\alpha+\mu h_\upsilon}^{\alpha+(\mu+1)h_\upsilon-1}\sum_{\psi=0}^{b-1}\sum_{j=\beta+\psi h_\eta}^{\beta+(\psi+1)h_\eta-1}\sum_{\varphi=0}^{d-1}\sum_{k=\gamma+\varphi h_\tau}^{\gamma+(\varphi+1)h_\tau-1}\right.\right.$$

$$\left.\left.\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}+\frac{1}{pqr}$$

$$=\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\sum_{i=\alpha+ch_\upsilon}^{\alpha+p-1}\sum_{j=\beta+bh_\eta}^{\beta+q-1}\sum_{j=\gamma+dh_\tau}^{\gamma+r-1}\left\|\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}. \tag{15}$$

Since $\{x_{ijk}\} \in [\mathcal{M},u,l^\infty,p,\|.,...,.\|]$ for all i,j,k, we have

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}<K,$$

From (14) and (15), we have

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}$$

$$\leq \quad \frac{1}{pqr}dcb.h_\upsilon h_\eta h_\tau\frac{\varepsilon}{2}+\frac{Kh_{\upsilon,\eta,\tau}}{pqr}$$

for $\frac{h_\upsilon h_\eta h_\tau}{pqr} \leq 1$, since $\frac{dcbh_\upsilon h_\eta h_\tau}{pqr} \leq 1$ and $\frac{Kh_{\upsilon,\eta,\tau}}{pqr}$ can be made less than $\frac{\varepsilon}{2}$, taking p, q, r sufficiently large so

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}<\varepsilon$$

for $\upsilon,\eta,\tau \geq \upsilon_0,\eta_0,\tau_0$, $\alpha \geq \alpha_0$, $\beta \geq \beta_0$, $\gamma \geq \gamma_0$.
Therefore, $[\mathcal{M},u,F_\theta,p,\|.,...,.\|] \cap [\mathcal{M},u,l^\infty,p,\|.,...,.\|] \subseteq [\mathcal{M},u,F,p,\|.,...,.\|]$.
(ii) Let $\{x_{ijk}\}=(-1)^{ijk}(ijk)^\psi$ where ψ is constant with $0<\psi<1$. Then

$$\sum_{i=\alpha}^{\alpha+h_\upsilon-1}\sum_{j=\beta}^{\beta+h_\eta-1}\sum_{j=\gamma}^{\gamma+h_\tau-1}x_{ijk}, \quad (\alpha,\ \beta,\ \gamma \geq 0)$$

Let $X=\mathbb{R}^n$. It is straight forward to verify that $\{x_{ijk}\} \in [\mathcal{M},u,F_\theta,p,\|.,...,.\|]$ with $\xi=0$. But $\{x_{ijk}\}$ is not bounded. □

Theorem 3. *The sequence space $[\mathcal{M}, u, [F], p, \|., ..., .\|]$ is a linear topological space total paranormed by*

$$g_3(x) = \sup_{\substack{p,q,r\geq 1,\ \alpha,\beta,\gamma\geq 1\\ 0\neq y_1,\dots,y_{n-1}\in X}} \left(\frac{1}{pqr} \sum_{i=\alpha}^{\alpha+p-1} \sum_{j=\beta}^{\beta+q-1} \sum_{k=\gamma}^{\gamma+r-1} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\| \frac{x_{i,j,k}}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right]^{p_{mno}} \right)$$

$$= \sup_{\substack{p,q,r\geq 1,\ \alpha,\beta,\gamma\geq 1\\ 0\neq y_1,\dots,y_{n-1}\in X}} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} u_{mno} M_{mno} \left[\left(h_{pqr\alpha\beta\gamma} \left(\left\| \frac{x_{i,j,k}}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right) \right]^{p_{mno}}.$$

Proof. Obviously $g_3(x) = 0 \Leftrightarrow x = 0$, $g_3(x) = g_3(-x)$ and g_3 is subadditive. Let $(x^{(m)})$ in $[\mathcal{M}, u, [F], p, \|., ..., .\|]$ such that $g_3(x^{(m)} - x) \to 0$ as $m \to \infty$ and (v_{mno}) be any sequence of scalars such that $v_{mno} \to v$ as $m, n, o \to \infty$. Since

$$g_3\left(x^{(m)}\right) \leq g_3(x) + g_3\left(x^{(m)} - x\right)$$

holds by subadditivity of $g_3, g_3\left(x^{(m)}\right)$ is bounded. Thus, we acquire

$$\begin{aligned}
& g_3\left(v_{mno}x^{(m)} - \nu x\right) \\
= & \sup_{\substack{p,q,r\geq 1,\ \alpha,\beta,\gamma\geq 1\\ 0\neq y_1,\dots,y_{n-1}\in X}} (\frac{1}{pqr} \sum_{i=\alpha}^{\alpha+p-1} \sum_{j=\beta}^{\beta+q-1} \sum_{k=\gamma}^{\gamma+r-1} \\
& \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} (\left\| \frac{v_{mno}x_{ijk}^{(m)} - vx_{ijk}}{\rho}, y_1, ..., y_{n-1} \right\|) \right]^{p_{mno}}) \\
\leq & |v_{mno} - v| \sup_{\substack{p,q,r\geq 1,\ \alpha,\beta,\gamma\geq 1\\ 0\neq y_1,\dots,y_{n-1}\in X}} (\frac{1}{pqr} \sum_{i=\alpha}^{\alpha+p-1} \sum_{j=\beta}^{\beta+q-1} \sum_{k=\gamma}^{\gamma+r-1} \\
& \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} (\left\| \frac{x_{i,j,k}^{(m)}}{\rho}, y_1, ..., y_{n-1} \right\|) \right]^{p_{mno}}) \\
& + |v| \sup_{\substack{p,q,r\geq 1,\ \alpha,\beta,\gamma\geq 1\\ 0\neq y_1,\dots,y_{n-1}\in X}} (\frac{1}{pqr} \sum_{i=\alpha}^{\alpha+p-1} \sum_{j=\beta}^{\beta+q-1} \sum_{k=\gamma}^{\gamma+r-1} \\
& \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} (\left\| \frac{x_{i,j,k}^{(m)} - x_{ijk}}{\rho}, y_1, ..., y_{n-1} \right\|) \right]^{p_{mno}}) \\
= & |v_{mno} - v| \, g_3\left(x^{(m)}\right) + |v| \, g_3\left(x^{(m)} - x\right) \to 0
\end{aligned}$$

as $m, n, o \to \infty$. This concludes the proof. $\square$

Definition 7. *A triple sequence $x=(x_{ijk})$ is named to be strongly p-Cesàro $\mathcal{I}$-summable $(0<p<\infty)$ to a limit ξ in $([\mathcal{M},[F],u,,p,\|.,...,.\|],g_3)$ if*

$$\left\{(i,j,k)\in\mathbb{N}^3:\frac{1}{mno}\sum_{i,j,k=1,1,1}^{m,n,o}\left(g_3\left(x_{ijk}-\xi e\right)\right)^p\geq\varepsilon\right\}\in\mathcal{I}_3$$

for every $\varepsilon>0$ and we write it as $x_{ijk}\to\xi\left[C,g_3\left(\mathcal{I}\right)\right]_p$.

Definition 8. *A triple sequence $x=(x_{ijk})$ is said to be g_3-ideal convergent to a number L in $([\mathcal{M},u,[F],p,\|.,...,.\|],g_3)$ if for each $\varepsilon>0$*

$$\left\{(i,j,k)\in\mathbb{N}^3:g_3\left(x_{ijk}-\xi e\right)\geq\varepsilon\right\}\in\mathcal{I}_3,$$

where

$$\begin{aligned}&g_3\left(x_{ijk}-\xi e\right)\\=&\sup_{\substack{p,q,r\geq1,\ \alpha,\beta,\gamma\geq1\\0\neq y_1,...,y_{n-1}\in X}}\left(\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\right.\\&\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left(\left\|\frac{x_{ijk}-\xi e}{\rho},y_1,...,y_{n-1}\right\|\right)\right]^{p_{mno}}.\end{aligned}$$

By $\mathcal{I}^3_{([\mathcal{M},u,[F],p,\|.,...,.\|],g_3)}$ we denote set of all $g_3(\mathcal{I})$-convergent sequences in $([\mathcal{M},u,[F],p,\|.,...,.\|],g_3)$.

Definition 9. *A triple sequence $x=(x_{ijk})$ is said to be g_3-ideal Cauchy in $([\mathcal{M},u,[F],p,\|.,...,.\|],g_3)$ if for every $\varepsilon>0$ there exist three numbers $P=P(\varepsilon)$, $Q=Q(\varepsilon)$, $R=R(\varepsilon)$ such that*

$$\left\{(i,j,k)\in\mathbb{N}^3:g_3\left(x_{ijk}-x_{PQR}\right)\geq\varepsilon\right\}\in\mathcal{I}_3.$$

Theorem 4. *If a triple sequence is ideal convergent in $([\mathcal{M},u,[F],p,\|.,...,.\|],g_3)$, then its limit is unique.*

Proof. For given $\varepsilon>0$ we define sets as:

$$K_1\left(\varepsilon\right)=\left\{(i,j,k)\in\mathbb{N}^3:g_3\left(x_{ijk}-\xi_1\right)\geq\frac{\varepsilon}{2}\right\}$$

and

$$K_2\left(\varepsilon\right)=\left\{(i,j,k)\in\mathbb{N}^3:g_3\left(x_{ijk}-\xi_2\right)\geq\frac{\varepsilon}{2}\right\}.$$

and suppose $g_3(\mathcal{I})-\lim x=\xi_1$ and $g_3(\mathcal{I})-\lim x=\xi_2$. Since $g_3(\mathcal{I})-\lim x=\xi_1$, we have $K_1(\varepsilon)\in\mathcal{I}_3$. Similarly, $g_3(\mathcal{I})-\lim x=\xi_2$ we have $K_2(\varepsilon)\in\mathcal{I}_3$. Now let $K(\varepsilon)=K_1(\varepsilon)\cup K_2(\varepsilon)$. Then $K(\varepsilon)\in\mathcal{I}_3$ and therefore, $K^c(\varepsilon)$ is a non-empty set and $K^c(\varepsilon)\in\mathcal{F}\left(\mathcal{I}_3\right)$. If $(i,j,k)\in\left(\mathbb{N}\times\mathbb{N}\times\mathbb{N}\right)\diagdown K\left(\varepsilon\right)$, then we have

$$g_3\left(\xi_1-\xi_2\right)\leq g_3\left(x_{ijk}-\xi_1\right)+g_3\left(x_{ijk}-\xi_2\right)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get $g_3(\xi_1 - \xi_2) = 0$ and hence $\xi_1 = \xi_2$. □

Theorem 5. *Let $x = (x_{ijk}) \in ([\mathcal{M}, u, [F], p, \|., ..., .\|], g_3)$ be ideal convergent to ξ iff there exists a set*

$$K = \{(i_p, j_q, k_r) \in \mathbb{N}^3 : i_1 < i_2 < ... < i_p < ...,\ j_1 < j_2 < ... < j_q < ...,\ k_1 < k_2 < ... < k_r < ...\}$$

with $K \in \mathcal{F}(\mathcal{I}_3)$ such that $g_3(x_{i_p j_q k_r} - \xi) \to 0$ as $i_p, j_q, k_r \to \infty$.

Proof. Let $g_3(\mathcal{I}) - \lim x = \xi$. Now write for $v = 1, 2, ...$

$$K_s(\varepsilon) = \left\{(p,q,r) \in \mathbb{N}^3 : g_3\left(x_{i_p j_q k_r} - \xi\right) \leq 1 + \frac{1}{v}\right\}$$

and

$$M_v(\varepsilon) = \left\{(p,q,r) \in \mathbb{N}^3 : g_3\left(x_{i_p j_q k_r} - \xi\right) > \frac{1}{v}\right\}.$$

Then $K_s \in \mathcal{I}_3$. Also

$$M_1 \supset M_2 \supset ... \supset M_i \supset M_{i+1} \supset ... \tag{16}$$

and

$$M_v \in \mathcal{F}(\mathcal{I}_3),\ v = 1, 2, ... \tag{17}$$

As we know $\{x_{i_p j_q k_r}\}$ is g_3-convergent to ξ. Assume $\{x_{ijk}\}$ is not g_3-convergent to ξ. Therefore, there is $\varepsilon > 0$ such that $g_3(x_{i_p j_q k_r} - \xi) \leq \varepsilon$ for infinitely many terms.

Let

$$M_\varepsilon = \left\{(p,q,r) \in \mathbb{N}^3 : g_3(x_{i_p j_q k_r} - \xi) > \varepsilon\right\}$$

and $\varepsilon > \frac{1}{v}$, $(v = 1, 2, ...)$. Then $M_\varepsilon \in \mathcal{I}_3$ and by (16) $M_v \subset M_\varepsilon$. Hence, $M_v \in \mathcal{I}_3$ which contradicts (17) and we get that $\{x_{ijk}\}$ is g_3-convergent to ξ.

Conversely, suppose that there exists a subset

$$K = \left\{(i_p, j_q, k_r) \in \mathbb{N}^3 : i_1 < ... < i_p < ...,\ j_1 < ... < j_q < ...,\ k_1 < ... < k_r < ...\right\}$$

with $K \in \mathcal{F}(\mathcal{I}_3)$ such that $g_3\text{-}\lim_{p,q,r\to\infty} x_{i_p j_q k_r} = \xi$ then there exists an $N(\varepsilon)$ such that

$$g_3(x_{ijk} - \xi) < \varepsilon \text{ for } i, j, k > N.$$

Let

$$K_\varepsilon = \left\{(i,j,k) \in \mathbb{N}^3 : g_3\left(x_{ijk} - \xi\right) \geq \varepsilon\right\}$$

and

$$K' = \{(i_{N+1}, j_{N+1}, k_{N+1}), (i_{N+2}, j_{N+2}, k_{N+2}), ...\}.$$

Then $K' \in \mathcal{F}(\mathcal{I}_3)$ and $K_\varepsilon \subseteq (\mathbb{N} \times \mathbb{N} \times \mathbb{N}) \diagdown K'$ which implise that $K_\varepsilon \in \mathcal{I}_3$. Hence $g_3(\mathcal{I}) - \lim x = \xi$. □

Theorem 6. *Let $g_3(\mathcal{I}) - \lim x = \xi_1$ and $g_3(\mathcal{I}) - \lim y = \xi_2$. Then*
(i) $g_3(\mathcal{I}) - \lim(x \pm y) = \xi_1 \pm \xi_2$
(ii) $g_3(\mathcal{I}) - \lim(\alpha x) = \alpha\xi_1$, $\alpha \in \mathbb{R}$.

Proof. It is easy to prove, so we omit it. □

Theorem 7. *If $0 < p < \infty$ and $x_{ijk} \to \xi\left[C, g_3(\mathcal{I})\right]_p$, then (x_{ijk}) is g_3-ideal convergent to ξ in $([\mathcal{M}, u, [F], p, \|., ..., .\|], g_3)$.*

Proof. Let $x_{ijk} \to \xi\left[C, g_3(\mathcal{I})\right]$. Then we have

$$\begin{aligned} \tfrac{1}{mno} \sum_{i,j,k=1,1,1}^{m,n,o} \left(g_3\left(x_{ijk} - \xi e\right)\right)^p &\geq \tfrac{1}{mno} \sum_{\substack{i,j,k=1 \\ \left(g_3\left(x_{ijk}-\xi e\right)\right) \geq \varepsilon}}^{m,n,o} \left(g_3\left(x_{ijk} - \xi e\right)\right)^p \\ &\geq \tfrac{\varepsilon^p}{mno} \left|\left\{(i,j,k) \in \mathbb{N}^3 : g_3\left(x_{ijk} - \xi e\right) \geq \varepsilon\right\}\right| \end{aligned}$$

and

$$\frac{1}{\varepsilon^p mno} \sum_{i,j,k=1,1,1}^{m,n,o} \left(g_3\left(x_{ijk} - \xi e\right)\right)^p \geq \frac{1}{mno} \left|\left\{(i,j,k) \in \mathbb{N}^3 : g_3\left(x_{ijk} - \xi e\right) \geq \varepsilon\right\}\right|$$

That is

$$\left\{(i,j,k) \in \mathbb{N}^3 : g_3\left(x_{ijk} - \xi e\right) \geq \varepsilon\right\} \in \mathcal{I}_3.$$

Hence (x_{ijk}) is g_3-ideal convergent to ξ in $([\mathcal{M}, u, [F], p, \|., ..., .\|], g_3)$. □

Theorem 8. *If $x = (x_{ijk})$ is $g_3(\mathcal{I})$-convergent to ξ in $([\mathcal{M}, [F], u, p, \|., ..., .\|], g_3)$ then $x_{ijk} \to \xi\left[C, g_3(\mathcal{I})\right]_p$.*

Proof. Suppose that $x = (x_{ijk})$ is g_3-ideal convergent to ξ in $([\mathcal{M}, [F], u, p, \|., ..., .\|], g_3)$. Then for $\varepsilon > 0$, we have $K_\varepsilon \in \mathcal{I}_3$, where $K_\varepsilon = \left\{(i,j,k) \in \mathbb{N}^3 : g_3\left(x_{ijk} - \xi e\right) \geq \varepsilon\right\}$. Since $x = (x_{ijk}) \in [\mathcal{M}, u, l^\infty, p, \|., ..., .\|]$, then there exists $K > 0$ such that

$$\left[u_{mno} M_{mno}\left(\left\|\frac{x_{i,j,k} - \xi e}{\rho}, y_1, ..., y_{n-1}\right\|\right)\right]^{p_{mno}} \leq K$$

for all i, j, k. Thus,

$$\begin{aligned} g_3\left(x_{ijk} - \xi e\right) = \sup_{\substack{p,q,r \geq 1,\ \alpha,\beta,\gamma \geq 1 \\ 0 \neq y_1,...,y_{n-1} \in X}} \Bigg(\frac{1}{pqr} \sum_{i=\alpha}^{\alpha+p-1} \sum_{j=\beta}^{\beta+q-1} \\ \sum_{k=\gamma}^{\gamma+r-1} \left[u_{mno} M_{mno}\left(\left\|\frac{x_{i,j,k} - \xi e}{\rho}, y_1, ..., y_{n-1}\right\|\right)\right]^{p_{mno}} \Bigg) \leq K. \end{aligned}$$

Hence

$$\begin{aligned} \tfrac{1}{mno} \sum_{i,j,k=1,1,1}^{m,n,o} \left(g_3\left(x_{ijk} - \xi e\right)\right)^p &= \tfrac{1}{mno} \sum_{\substack{i,j,k=1,1,1 \\ i,j,k \notin K_\varepsilon}}^{m,n,o} \left(g_3\left(x_{ijk} - \xi e\right)\right)^p \\ &+ \tfrac{1}{mno} \sum_{\substack{i,j,k=1,1,1 \\ i,j,k \in K_\varepsilon}}^{m,n,o} \left(g_3\left(x_{ijk} - \xi e\right)\right)^p \leq \varepsilon^p + \tfrac{K^p}{mno} \left|K_\varepsilon\right|. \end{aligned}$$

Then the set K_ε on the right hand side of above inequality belongs to $\mathcal{I}_3$. Therefore, $x_{ijk} \to \xi\left[C, g_3(\mathcal{I})\right]_p$. Hence the proof is concluded. □

Let X and Y be two triple sequence spaces. We use the notation $X_{reg} \subset Y_{reg}$ to mean if the triple sequence x converges to a limit ξ in X then the sequence x converges to the same limit in Y.

Theorem 9. $\left(\mathcal{I}^3_{([\mathcal{M},u,[F],p,\|.,...,.\|],g_3)}\right)_{reg} = \left([C, g_3(\mathcal{I})]_p\right)_{reg}$.

Proof. By combining Theorem (7) and Theorem (8) we have the proof. □

Theorem 10. *Let a complete paranormed space be* $([\mathcal{M}, u, [F], p, \|., ..., .\|], g_3)$. *Then a sequence in* $[\mathcal{M}, u, [F], p, \|., ..., .\|]$ *is* g_3*-ideal convergent iff it is* g_3*-ideal Cauchy.*

Proof. Let $g_3(\mathcal{I}_3) - \lim x = \xi$. Then, we get $X(\varepsilon) \in \mathcal{I}_3$, where

$$X(\varepsilon) = \left\{(i,j,k) \in \mathbb{N}^3 : g_3(x_{ijk} - \xi) \geq \frac{\varepsilon}{2}\right\}.$$

This implies

$$X^c(\varepsilon) = \left\{(i,j,k) \in \mathbb{N}^3 : g_3(x_{ijk} - \xi) < \frac{\varepsilon}{2}\right\} \in \mathcal{F}(\mathcal{I}_3).$$

Let $m, n, o \in X^c(\varepsilon)$. Then $g_3(x_{mno} - \xi) < \frac{\varepsilon}{2}$. Now let

$$Y(\varepsilon) = \left\{(i,j,k) \in \mathbb{N}^3 : g_3(x_{ijk} - \xi) \geq \varepsilon\right\}.$$

We need to demonstrate that $Y(\varepsilon) \subset X(\varepsilon)$. Let $(i,j,k) \in Y(\varepsilon)$. Then $g_3(x_{mno} - x_{ijk}) \geq \varepsilon$ and therefore $g_3(x_{ijk} - \xi) \geq \varepsilon$, that is $(i,j,k) \in X(\varepsilon)$. Otherwise, if $g_3(x_{ijk} - \xi) < \varepsilon$ then

$$\varepsilon \leq g_3(x_{ijk} - x_{mno}) \leq g_3(x_{ijk} - \xi) + g_3(x_{mno} - \xi) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which is not possible. Thus, $Y(\varepsilon) \subset X(\varepsilon)$ and therefore, $x = (x_{ijk})$ is g_3-ideal convergent sequences.

Conversely, let $x = (x_{ijk})$ is g_3-ideal Cauchy but not g_3-ideal convergent sequences. Then there exist $(t', w', v') \in \mathbb{N}^3$ such that

$$D(\varepsilon) = \left\{(i,j,k) \in \mathbb{N}^3 : g_3(x_{ijk} - x_{t'w'v'}) \geq \varepsilon\right\} \in \mathcal{I}_3$$

and $G(\varepsilon) \in \mathcal{I}_3$, where

$$G(\varepsilon) = \left\{(i,j,k) \in \mathbb{N}^3 : g_3(x_{ijk} - \xi) < \frac{\varepsilon}{2}\right\},$$

that is, $G^c(\varepsilon) \in \mathcal{F}(\mathcal{I}_3)$, since $g_3(x_{ijk} - x_{mno}) \leq 2g_3(x_{ijk} - \xi) < \varepsilon$. If $g_3(x_{ijk} - \xi) < \frac{\varepsilon}{2}$ then $D^c(\varepsilon) \in \mathcal{I}_3$, that is, $D(\varepsilon) \in \mathcal{F}(\mathcal{I}_3)$ which leads to a contradiction. Hence, the result is obtained. □

Author Contribution Statements The authors contributed equally to this work. All authors read and approved the final copy of this paper.

Declaration of Competing Interests The authors declare that they have no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

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