http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 76–103 (2024)
DOI:10.31801/cfsuasmas.1147449
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS
SERIES A1

Research Article; Received: July 23, 2022; Accepted: October 6, 2023

EXPONENTIATED GENERALIZED RAMOS-LOUZADA DISTRIBUTION WITH PROPERTIES AND APPLICATIONS

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Abstract. In this paper, we propose a new generalization of Ramos-Louzada (RL) distribution based on two additional shape parameters. Along with the genesis of its distributional form, the derivation of cumulative density function (cdf), survival and hazard rate functions, the quantile function (qf), moments, moment generating function (mgf), Shannon and Renyi entropies, order statistics and a linear representation of the proposed distribution are inspected. Several estimation methods of the model parameters are discussed throughout two comprehensive simulation studies conducted to compare its performance against some lifetime distributions. Application of a real dataset is presented to illustrate the potentiality of this distribution in line with the simulation studies.

1. Introduction

Lifetime modeling of complex studies has created a growing interest in the generation of flexible distributions that can provide solutions to certain problems of lifetime systems. Ramos-Louzada is such a distribution recently proposed by Ramos and Louzada ([24]) to take instantaneous failures into account that can inevitably occur in many lifetime applications. It is announced to be a worthwhile alternative to the Exponential and Lindley ([19]) distributions and take the forms of both with a shape parameter $\lambda \geq 2$. That is, the distribution becomes the exponential distribution for large values of λ and it resembles to the Lindley distribution as λ decreases towards 2.

2020 *Mathematics Subject Classification.* 05C38, 15A15, 05A15, 15A18.
Keywords. Lifetime datasets, Ramos-Louzada distribution, exponentiated generalized class, EGRL distribution, information criteria.
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Let the random variable X follows the RL distribution with the rate parameter $\theta = \frac{1}{\lambda}$, $\lambda \geq 2$;

$$g(x) \quad = \quad \frac{(\theta^2 x - 2\theta + 1)\theta}{1-\theta}\; e^{-\theta x}, \tag{1}$$

where $x \geq 0$ and $0.5 \geq \theta > 0$. The cdf of $X \sim RL(\theta)$ is defined as

$$G(x) \quad = \quad 1 - \frac{(\theta^2 x - \theta + 1)}{1-\theta}\; e^{-\theta x}. \tag{2}$$

Although the RL distribution is attractive for its simplicity, it fails to provide precise evaluation of many lifetime datasets, since it contains only one parameter. Many researchers benefit from generalizing baseline (stated otherwise parent or target) distributions by adding one or more parameters into the model to increase the model fit and overcome the absence of sufficient flexibility in modeling the data. In this respect, Al-Mofleh et al. ([3]) recently proposed a two-parameter generalization of the RL distribution by inserting a power parameter into the model. They showed that the generalized Ramos-Louzada (GRL) distribution performs better than some well-known distributions such as Marshall-Olkin ([21]), exponentiated exponential ([11]) and generalized Lindley ([23]) distributions with respect to some bias and accuracy measures.

This paper proposes a new three-parameter model as a competetive extension for this generalization of the RL distribution, namely the exponentiated generalized Ramos-Louzada (EGRL) distribution. The new distribution relies on the class of distributions established by Corderio et al. ([7]). The usual definition of the probability density function (pdf) of this family of distributions is

$$f(x) = \alpha\beta g(x)\Big[1 - G(x)\Big]^{\alpha-1}\left(1 - \Big[1 - G(x)\Big]^{\alpha}\right)^{\beta-1}, \tag{3}$$

where $\alpha > 0$ and $\beta > 0$ are two shape parameters and $g(x)$ and $G(x)$ are the pdf and cdf of the baseline distribution, respectively. The shape parameters α and β in equation (3) provide better flexibility in the tails of the data and increase the entropy in the center ([7, p. 2]). The cdf of the family of distributions is of the form

$$F(x) = \left(1 - \Big[1 - G(x)\Big]^{\alpha}\right)^{\beta}. \tag{4}$$

Our basic motivation for such generalization is to provide a better fit of RL distribution to the wider range of problems in statistics. It is also of our goal to achieve reliable estimation of model parameters considering various estimation methods. This is particularly important as it affects the model selection process. Evaluation of model fit via goodness of fit statistics is a usual practice in the literature. Al-Mofleh et al. ([3]) consider only the minus log likelihood (-ℓ), Cramer-von Mises (C*; [9]) and Kolmogorov-Smirnov (KS*; [16, 31]) goodness of fit statistics. The assessment of model fit via these goodness of fit statistics might produce biased results, since they do not take the model complexity into account when choosing the best distribution in a set of distributions. The distributions under consideration should also be compared to each other by means of using information criteria.

Incorporating additional adequate parameter(s) into the model improves the model fit and provides more flexibility in analyzing datasets. However, caution should be taken when generalizing baseline distributions using more parameters in the model. Achieving a good model fit requires taking into account the balance between the sample size and the number of parameters in the model (bias versus variance tradeoff as used in the literature). The information criteria such as Akaike information criterion (AIC; [1, 2]) and Bayesian information criterion (BIC; [29]) are originally developed for solving this problem as they do not only rely on log likelihood values, but also on penalty values. The log likelihood represents the fit of a model to the data at hand as the penalty value penalizes the model depending on (a function of) the number of parameters in the model. Thus, we compare the EGRL distribution against a set of alternative distributions by means of not only using model fit statistics, but also different types of information criteria.

The outline of the paper is as follows. In Section 2, we derive various statistical and reliability properties of the EGRL distribution. In Section 3, we elaborate on the methods of maximum likelihood estimation (MLE), least squares estimation (LSE), weighted least squares estimation (WLSE), and Cramer-von Mises estimation (CVME) to obtain the estimates of model parameters and their standard errors for the EGRL distribution. In Section 4, we perform two simulation studies. In the first simulation study, we evaluate the performance of the methods in estimating the parameters of the EGRL distribution with respect to bias, precision, and accuracy measures. In the second simulation study, we compare the performance of the EGRL distribution to that of a set of other lifetime distributions with respect to some goodness of fit statistics and information criteria for each estimation method. In Section 5, we exemplify the applicability of EGRL distribution for a real life problem. We illustrate that the goodness of fit statistics may not be able to detect the best distribution in a set of distributions and information criteria should be used instead when comparing the performance of distributions. The paper will be concluded with a short discussion.

2. The EGRL Distribution

2.1. **Probability density and cumulative density functions.** Incorporating equations (1) and (2) into the general definition in equation (3), we obtain the pdf of EGRL distribution which is given by

$$f(x) = \alpha\beta \; \frac{(\theta^2 x - 2\theta + 1)\theta}{1-\theta} \; e^{-\theta x} \left[\frac{(\theta^2 x - \theta + 1)}{1-\theta} \; e^{-\theta x} \right]^{\alpha - 1}$$

$$\times\left(1-\left[\frac{(\theta^2x-\theta+1)}{1-\theta}\ e^{-\theta x}\right]^{\alpha}\right)^{\beta-1}. \tag{5}$$

For $\alpha=\beta=1$, the distribution reduces to the RL distribution. Similarly, by replacing $G(x)$ in equation (4) with the cdf of RL distribution in equation (2), we obtain the cdf of EGRL distribution as

$$F(x) \quad = \quad \left(1-\left[\frac{(\theta^2x-\theta+1)e^{-\theta x}}{1-\theta}\right]^{\alpha}\right)^{\beta}. \tag{6}$$

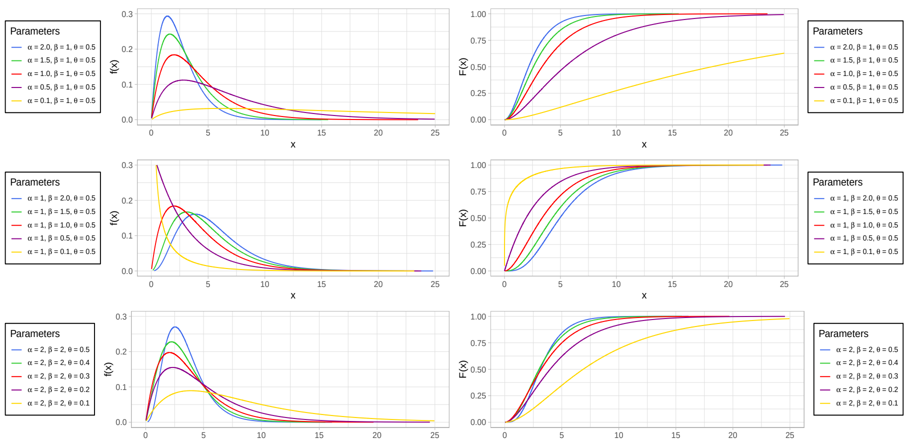

FIGURE 1. The pdf and cdf plots of EGRL distribution with varying values of α, β and θ parameters.

Figure 1 displays the plots for the pdf and cdf of EGRL distribution using different values of α, β and θ parameters. As can be seen on the left panel of the figure, the EGRL distribution is flexible in the sense that it can be positively skewed with or without reversed-J shape. The plots on the right panel of the figure show that the cdf of EGRL distribution increases towards one with increasing values of the random variable X for varying values of parameters α, β and θ.

2.2. **Survival and hazard rate functions.** The survival function (stated otherwise reliability function) is commonly used for lifetime datasets which often represents the probability of a patient's survival or an object's resistance until a predetermined time point. The survival function of EGRL distribution indicating the complement of the cdf in equation (6) is given by

$$S(x) \quad = \quad 1-\left(1-\left[\frac{(\theta^2x-\theta+1)e^{-\theta x}}{1-\theta}\right]^{\alpha}\right)^{\beta}. \tag{7}$$

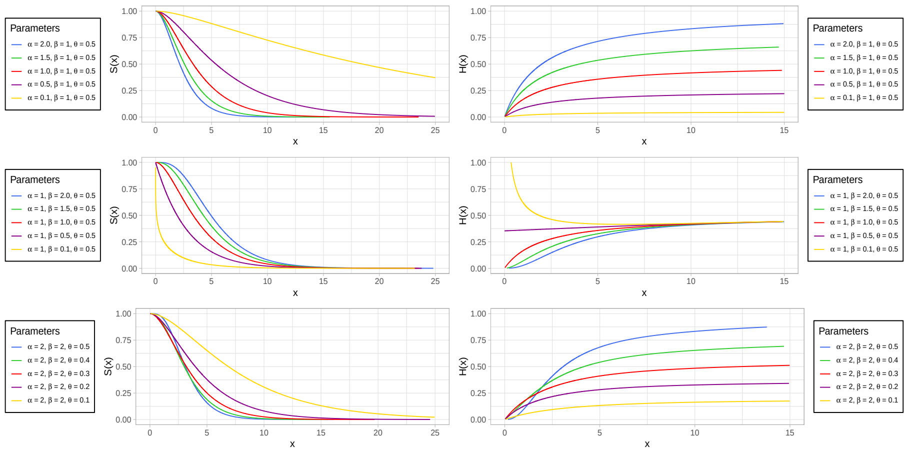

FIGURE 2. The survival and hazard rate plots of EGRL distribution with varying values of α, β and θ parameters.

Another widely used tool that can serve to characterize the EGRL distribution is the hazard rate function which indicates the probability of the occurence of an event. The values of hazard rate function for the EGRL distribution can easily be obtained by

$$H(x) = \frac{f(x)}{S(x)}, \tag{8}$$

where $f(x)$ is the pdf in equation (5) and $S(x)$ is the survival function in equation (7).

Figure 2 displays the plots of survival and hazard rate functions for the EGRL distribution. These plots exhibit increasing, decreasing, and reversed-J shaped hazard rate functions and decreasing survival functions with increasing values of random variable X.

2.3. **The quantile function.** The quantile function (qf) of EGRL distribution is the inverse of the cdf in equation (6). By applying $w = -\theta x - \frac{1-\theta}{\theta}$ transformation and using w in Lambert form we^w for $u = G(x)$ in equation (2), we obtain

$$we^w = \frac{(1-\theta)(u-1)e^{1-\frac{1}{\theta}}}{\theta}. \tag{9}$$

This means that w can be defined as a Lambert function of the real argument we^w. The real argument $we^w \in (-\frac{1}{e}, 0)$ for $u \in (0,1)$. Thus,

$$Q_{EGRL}(u) = \frac{-\theta W_{-1}\left[\frac{(\theta-1)\left[(1-u^{\frac{1}{\beta}})^{\frac{1}{\alpha}}\right]e^{1-\frac{1}{\theta}}}{\theta}\right] + \theta - 1}{\theta^2}, \tag{10}$$

where W_{-1} is the negative branch of the Lambert function, $0 < \theta \leq 0.5$, and $0 < u < 1$ (see [7, p. 2]). The values of the negative branch of Lambert function W_{-1} can easily be obtained using `lambertWm1` subroutine of `lamW` package in `R` statistical software.

The Bowley skewness ([15]) and Moorsis kurtosis ([22]) measures for EGRL distribution are defined by

$$B = \frac{Q_{EGRL}(3/4) + Q_{EGRL}(1/4) - 2Q_{EGRL}(2/4)}{Q_{EGRL}(3/4) - Q_{EGRL}(1/4)} \tag{11}$$

and

$$M = \frac{Q_{EGRL}(3/8) - Q_{EGRL}(1/8) + Q_{EGRL}(7/8) - Q_{EGRL}(5/8)}{Q_{EGRL}(6/8) - Q_{EGRL}(2/8)}, \tag{12}$$

where, for example, $Q_{EGRL}(3/4)$ is the third quartile and $Q_{EGRL}(5/8)$ is the fifth octile of the qf for the EGRL distribution. Table 1 shows how these measures behave with varying values of parameters α, β and θ. In line with the pdf plots in Figure 1, increasing values of Moorsis kurtosis measure are associated with the pdfs with heavier tails. Positive values of the Bowley skewness measure in this table indicate that the distributions are right skewed.

TABLE 1. The Bowley skewness and Moorsis kurtosis measures with varying values of α, β, and θ parameters.

α	β	θ	Bowley skewness	Moorsis kurtosis
0.1	1.0	0.5	0.23	1.25
0.5	1.0	0.5	0.19	1.30
1.0	1.0	0.5	0.17	1.32
1.5	1.0	0.5	0.15	1.33
2.0	1.0	0.5	0.15	1.34
1.0	0.1	0.5	0.77	2.49
1.0	0.5	0.5	0.24	1.16
1.0	1.5	0.5	0.14	1.46
1.0	2.0	0.5	0.13	1.56
2.0	2.0	0.1	0.18	1.31
2.0	2.0	0.2	0.17	1.32
2.0	2.0	0.3	0.15	1.33
2.0	2.0	0.4	0.13	1.41
2.0	2.0	0.5	0.12	1.61

2.4. **Moments.** We follow an analogous procedure to the one given in the previous subsection with a slightly different transformation

$$v = 1 - G(x) = \frac{(\theta^2 x - \theta + 1)}{1 - \theta} \; e^{-\theta x}, \tag{13}$$

where $0 < v < 1$. The mth moment of EGRL distribution is given by

$$E(X^m) = \int_0^\infty x^m f(x)dx = \alpha\beta \int_0^1 (-1)^{m+1} \big[z(v)\big]^m v^{\alpha-1}(1-v^\alpha)^{\beta-1}dv, \quad (14)$$

where

$$z(v) = \frac{-\theta W_{-1}\left[\frac{(1-\theta)ve^{1-\frac{1}{\theta}}}{\theta}\right] + \theta - 1}{\theta^2}.$$

The next subsection recalls some useful definitions and power series expansions that can be used to obtain the moments, mgf, Shannon and Renyi entropies, and order statistics of EGRL distribution.

2.5. **Useful definitions and power series expansions.** Let T be a random variable from the exponentiated exponential distribution which has the following pdf

$$r(t) = \alpha\beta e^{-\alpha t}(1-e^{-\alpha t})^{\beta-1}, \quad (15)$$

and cdf

$$R(t) = (1-e^{-\alpha t})^\beta, \quad (16)$$

where $\alpha,\ \beta,\ t > 0$ ([11]). The pdf of a random variable from the exponentiated generalized family of distributions can also be defined as

$$f(x) = \frac{g(x)}{1-G(x)} r\bigg(-\log\big[1-G(x)\big]\bigg), \quad (17)$$

where $T = -\log\big[1-G(X)\big]$ follows the exponentiated exponential distribution ([4]). By using equation (17), $u = G(x)$, and the power series expansion

$$-\log(1-u) = \sum_{i=0}^{\infty} \frac{u^{i+1}}{i+1}, \quad (18)$$

the pdf $r\bigg(-\log\big[1-G(x)\big]\bigg)$ becomes

$$r\bigg(-\log\big[1-G(x)\big]\bigg) = r\bigg(\sum_{i=0}^{\infty}\frac{u^{i+1}}{i+1}\bigg) = \alpha\beta e^{-\alpha D}(1-e^{-\alpha D})^{\beta-1}, \quad (19)$$

where $D = \sum_{i=0}^{\infty} \frac{u^{i+1}}{i+1}$. Similarly, by using equation (18), $u = G(x)$, and the power series expansion above, the corresponding cdf is defined as

$$R\bigg(-\log\big[1-G(x)\big]\bigg) = R\bigg(\sum_{i=0}^{\infty}\frac{u^{i+1}}{i+1}\bigg) = (1-e^{-\alpha D})^\beta. \quad (20)$$

By applying another useful power series expansion

$$(1-y)^a = \sum_{k=0}^{\infty}\binom{a}{k}(-1)^k|y|, |y| < 1,$$

we obtain

$$[1-F(x)]^{n-r}=\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^jF^j(x), \tag{21}$$

which in turn will be used particularly to establish the order statistics for the EGRL distribution.

2.6. **Moments using power series expansions and quantile function.** By using equations (19) and (21), the mth moment of EGRL distribution can also be defined as follows:

$$\begin{aligned}E(X^m)=&\int_0^\infty x^m\frac{g(x)}{1-G(x)}r\Big(-\log\big[1-G(x)\big]\Big)dx=\int_0^1 Q^m(u)\frac{1}{1-u}r\Big(\sum_{i=0}^\infty\frac{u^{i+1}}{i+1}\Big)du\\ =&\alpha\beta\int_0^1 Q^m(u)\frac{1}{1-u}e^{-\alpha D}(1-e^{-\alpha D})^{\beta-1}du,\end{aligned} \tag{22}$$

where $D=\sum_{i=0}^\infty\frac{u^{i+1}}{i+1}$. Here, $Q(u)=G^{-1}(u)=x$ is the quantile function (i.e., the inverse of the cdf) in equation (9), so that $u=G(x)$ and $du=g(x)dx$.

2.7. **Moment generating function.** Following the procedure used to obtain the moments in the previous subsection, the mgf of EGRL distribution can be obtained as

$$\begin{aligned}E(e^{bX})=&\int_0^\infty e^{bx}\frac{g(x)}{1-G(x)}r\Big(-\log\big[1-G(x)\big]\Big)dx=\int_0^1 e^{bQ(u)}\frac{1}{1-u}r\Big(\sum_{i=0}^\infty\frac{u^{i+1}}{i+1}\Big)du\\ =&\alpha\beta\int_0^1 e^{bQ(u)}\frac{1}{1-u}e^{-\alpha D}(1-e^{-\alpha D})^{\beta-1}du.\end{aligned} \tag{23}$$

2.8. **Shannon entropy.** The Shannon entropy ([30]) is a measure to ascertain the information provided by a random variable. The Shannon entropy for the random variable X from the EGRL distribution is given by

$$\eta_S=-E\Bigg(\log\bigg[\frac{g(x)r(t)}{1-G(x)}\bigg]\Bigg), \tag{24}$$

where $r(t)$ is used as a generator to attain the family of distributions in equation (3).

The association between the Shannon entropy for the generator variable T which has the support $[0,\infty]$ and the variable X from the beta-exponential-X family of distributions can be defined by using $T=-\log\big[1-G(X)\big]$, and thus, $X=G^{-1}(1-e^{-T})$ ([4]). This association also applies to our case for which the variable T is from the exponentiated exponential distribution which has the support $[0,\infty]$ and the variable X is from the EGRL distribution, since the exponentiated generalized family of distributions is a special case of the beta-exponential-X family of distributions. Thus, the Shannon entropy above is defined as

$$\begin{aligned}\eta_S =& -E\bigg(\log f\big[G^{-1}(1-e^{-T})\big]\bigg)+\eta_T-\mu_T,\\ =& -E\bigg(\log f\big[G^{-1}(1-e^{-T})\big]\bigg)\\ &+\log\big[(\alpha\beta)^{-1}\big]+\beta\Psi(\beta+1)-(\beta-1)\Psi(\beta)-\Psi(1)\\ &+\frac{\Psi(\beta+1)-\Psi(1)}{\alpha},\end{aligned} \tag{25}$$

where $\eta_T=\log\big[(\alpha\beta)^{-1}\big]+\beta\Psi(\beta+1)-(\beta-1)\Psi(\beta)-\Psi(1)$ is the Shannon entropy for random variable T, $\mu_T=\frac{\Psi(\beta+1)-\Psi(1)}{\alpha}$ is its mean and $\Psi(.)$ is the digamma function [4, p. 68].

2.9. **Renyi entropy.** The Renyi entropy ([26]) is another widely used measure to quantify the information in random variables. The Renyi entropy is an extension of the Shannon entropy. The Renyi entropy of order γ for the random variable X from the EGRL distribution is given by

$$\begin{aligned}\eta_R =&\frac{1}{1-\gamma}\log\int_0^\infty f^\gamma(x)dx=\frac{1}{1-\gamma}\log\int_0^\infty\frac{g^\gamma(x)}{1-G^\gamma(x)}r^\gamma\bigg(-\log\big[1-G(x)\big]\bigg)dx\\ =&\frac{1}{1-\gamma}\log\int_0^1\frac{g^{\gamma-1}\big[Q(u)\big]}{1-u^\gamma}r^\gamma\bigg(\sum_{i=0}^{\infty}\frac{u^{i+1}}{i+1}\bigg)du\\ =&\frac{\alpha^\gamma\beta^\gamma}{1-\gamma}\log\int_0^1\frac{g^{\gamma-1}\big[Q(u)\big]}{1-u^\gamma}e^{-\alpha\gamma D}(1-e^{-\alpha D})^{\gamma(\beta-1)}du,\end{aligned} \tag{26}$$

where $g(x)$ is the pdf of RL distribution and $D=\sum_{i=0}^{\infty}\frac{u^{i+1}}{i+1}$. The value of order γ influences the information obtained from random variable X. The Renyi entropy recovers the minimum entropy if $\gamma=\infty$, the maximum entropy if $\gamma=0$, and Shannon's entropy if $\gamma\to1$ ([27]).

2.10. **Order statistics.** Let $X_{(1)}=\min\{X_1,X_2,...,X_n\}$ and $X_{(n)}=\max\{X_1,X_2, ...,X_n\}$ are the smallest and largest values of a random sample $X_1,X_2,...,X_n$, respectively. In line with Arnold et al. ([5]), the pdf of the rth order statistic (i.e, the rth smallest value) is defined by

$$f_{(r)}(x)=\binom{n}{r}F^{r-1}(x)[1-F(x)]^{n-r}f(x). \tag{27}$$

By applying the power series expansion in equation (23), the pdf of the rth order statistic is defined by

$$f_{(r)}(x)=\binom{n}{r}F^{r-1}(x)[1-F(x)]^{n-r}f(x)$$

$$
\begin{aligned}
&=\frac{n!}{(r-1)!(n-r)!}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j F^{j+r-1}(x)f(x)\\
&=\frac{n!}{(r-1)!(n-r)!}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j R^{j+r-1}\Big(-\log\big[1-G(x)\big]\Big)\\
&\times\frac{g(x)}{1-G(x)}r\Big(-\log\big[1-G(x)\big]\Big)\\
&=\frac{n!}{(r-1)!(n-r)!}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j R^{j+r-1}\Big(\sum_{i=0}^{\infty}\frac{u^{i+1}}{i+1}\Big)\\
&\times\frac{g(x)}{1-G(x)}r\Big(\sum_{i=0}^{\infty}\frac{u^{i+1}}{i+1}\Big)\\
&=\frac{n!}{(r-1)!(n-r)!}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j(1-e^{-D})^{\beta(j+r-1)}\\
&\times\frac{g(x)}{1-G(x)}e^{-\alpha D}(1-e^{-\alpha D})^{\beta-1},
\end{aligned}
\tag{28}
$$

where $g(x)$ and $G(x)$ are the pdf and cdf of RL distribution and $D=\sum_{i=0}^{\infty}\frac{u^{i+1}}{i+1}$.

2.11. **Linear representation.** Corderio and Lemonte ([8]) produce the linear representations of equations (3) and (4). We summarize their procedure here by using $G(x)$ as the cdf of the baseline RL distribution. By applying the generalized binomial expansion in equation (23) twice in equation (4), the cdf of EGRL distribution is defined as

$$
\begin{aligned}
F(x)=&\Big(1-\Big[1-G(x)\Big]^{\alpha}\Big)^{\beta}=\sum_{k=0}^{\infty}(-1)^k\binom{\beta}{k}\big[1-G(x)\big]^{\alpha k}\\
=&\sum_{k=0}^{\infty}(-1)^k\binom{\beta}{k}\sum_{j=0}^{\infty}(-1)^j\binom{\alpha k}{j}G^j(x)=\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}(-1)^{j+k+1}\binom{\beta}{k}\binom{\alpha k}{j+1}G^{j+1}(x),
\end{aligned}
$$

where $G^{j+1}(x)$ is the cdf of RL distribution with a power parameter $j+1$. In other words, the cdf of EGRL distribution can be defined as a linear combination of the cdfs of RL distributions. By taking the derivative of $G^{j+1}(x)$ with respect to $x\geq 0$, we obtain the linear representation of the pdf of EGRL distribution which is given by

$$
f(x)=\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}(-1)^{j+k+1}\binom{\beta}{k}\binom{\alpha k}{j+1}(j+1)g(x)G^j(x),
$$

where $g(x)$ is the pdf and $G(x)$ is the cdf of the baseline RL distribution.

3. Parameter Estimation

In this section, we present the parameter estimation procedure by means of four methods: maximum likelihood estimation (MLE), least squares estimation (LSE), weighted least squares estimation (WLSE), and Cramer von Mises estimation (CVME). The LSE, WLSE, and CVME methods are included in the study as an alternative to MLE due to their ease of use.

3.1. **Maximum likelihood estimation.** The log likelihood function of a random sample $X = (X_1, X_2, ..., X_n)$ from the EGRL distribution is given by

$$\begin{aligned}\ell =& n\log(\alpha) + n\log(\beta) + n\log(\theta) - n\alpha\log(1-\theta) + \sum_{i=1}^{n}\log(\theta^2 x_i - 2\theta + 1)\\ &-\alpha\theta\sum_{i=1}^{n}x_i + (\alpha-1)\sum_{i=1}^{n}\log(\theta^2 x_i - \theta + 1)\\ &+(\beta-1)\sum_{i=1}^{n}\log\left(1 - \left[\frac{(\theta^2 x_i - \theta + 1)}{1-\theta}e^{-\theta x_i}\right]^{\alpha}\right).\end{aligned} \tag{29}$$

The elements of score vector $(\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\beta}, \frac{\partial\ell}{\partial\theta})$ containing the first derivatives (stated otherwise the gradients) of the log likelihood function with respect to parameters α, β and θ are given below.

$$\begin{aligned}\frac{\partial\ell}{\partial\alpha} =& \frac{n}{\alpha} - n\log(1-\theta) - \theta\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\log(\theta^2 x_i - \theta + 1)\\ &+(1-\beta)\sum_{i=1}^{n}\frac{\zeta_i^{\alpha}\left(\log(\zeta_i) - \theta x_i\right)}{e^{\alpha\theta x_i} - \zeta_i^{\alpha}},\\ \frac{\partial\ell}{\partial\beta} =& \frac{n}{\beta} + \sum_{i=1}^{n}\log\left(1 - \left(\zeta_i e^{-\theta x_i}\right)^{\alpha}\right),\\ \frac{\partial\ell}{\partial\theta} =& \frac{n}{\theta} + \frac{n\alpha}{1-\theta} + \sum_{i=1}^{n}\frac{2(\theta x_i - 1)}{\theta^2 x_i - 2\theta + 1} - \alpha\sum_{i=1}^{n}x_i + (\alpha-1)\sum_{i=1}^{n}\frac{2\theta x_i - 1}{\theta^2 x_i - \theta + 1}\\ &+(1-\beta)\sum_{i=1}^{n}\frac{\alpha x_i\zeta_i^{\alpha}\left[\theta^3 x_i - \theta^2(x_i+2) + 4\theta - 1\right]}{(1-\theta)(\theta^2 x_i - \theta + 1)(e^{\alpha\theta x_i} - \zeta_i^{\alpha})}.\end{aligned} \tag{30}$$

where $\zeta_i = \frac{(\theta^2 x_i - \theta + 1)}{1-\theta}$.

Maximum likelihood estimates (MLEs) are described analytically by setting the elements of the score vector equal to zero and solving for each parameter. The resulting equations $\frac{\partial\ell}{\partial\alpha} = 0$, $\frac{\partial\ell}{\partial\beta} = 0$ and $\frac{\partial\ell}{\partial\theta} = 0$ need to be solved simultaneously. Maximizing the log likelihood function with respect to parameters α, β and θ can be performed using a reliable non-linear optimization technique such as Nelder and

Mead (NM) or the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm. Maximizing the log likelihood (or minimizing minus the log likelihood) function can be achieved by using `maxLik` and `optim` subroutines of `maxLik` and `stats` packages in `R` statistical software.

When maximizing the log likelihood function above, the initials of α, β, and θ parameters must be specified. To easily obtain the initial for parameter θ by means of using the usual RL distribution in equation (1), the initials of parameters α and β were set to 1. The initial of parameter θ was obtained by taking the inverse of the root obtained in $\mu = \frac{\lambda^2}{\lambda - 1}$ (see [24, p. 250]), where parameter μ was replaced by the sample mean $\bar{x}$. The resulting initial for this parameter is $\theta_{\text{init}} = \frac{2}{\bar{x} + \sqrt{\bar{x}^2 - 4\bar{x}}}$.

We do not provide the analytical expressions of the entries of Hessian matrix for the log likelihood function of EGRL distribution which are too complicated. The standard errors of model parameters can be obtained by an approximate Hessian matrix using the default option (i.e., the finite-difference approach) in the `maxLik` package. The square root of diagonals for the inverse of minus the Hessian matrix gives the standard errors of parameter estimates. However, the same standard errors can be obtained by using `summary` function in the `maxLik` package. Note that this approximation technique does not always converge for the standard errors of model parameters. In such a case, nonparametric bootstrapping (NB; [10]) is a reasonable alternative to estimate the standard errors of parameters. The NB can also be used to obtain an estimate of bias to compare the performance of estimation methods presented in this paper, which will be evaluated in the application section.

The estimates of model parameters and their standard errors can also be obtained by maximizing a function of the cdf of EGRL distribution or a weighted form of this function known as the method of (weighted) least squares estimation which will be presented in the next subsection.

3.2. **The method of (Weighted) Least-squares estimation.** Based on Swain et al. ([33]), the least squares estimates of model parameters and their standard errors can be attained by maximizing

$$-\sum_{i=1}^{n}\left[F(x_{(i)}) - \frac{i}{n+1}\right]^2, \tag{31}$$

where $F(x_{(i)})$ is the cdf of the ordered random variables $x_{(1)} < x_{(2)} < ... < x_{(n)}$, see also [28, p. 181]. Thus, the least squares estimates for the EGRL distribution are obtained by maximizing

$$-\sum_{i=1}^{n}\left[\left(1 - \left[1 - G(x_{(i)})\right]^{\alpha}\right)^{\beta} - \frac{i}{n+1}\right]^2, \tag{32}$$

where $F(x_{(i)})$ in equation (33) is replaced by the cdf of the ordered random variables for the EGRL distribution. Here, $G(x_{(i)})$ represents the cdf of the ordered random variables for the baseline RL distribution in equation (2).

The weighted least squares estimation (WLSE) can be more reliable than the usual least squares estimation (LSE) when the data involve heteroscedasticity which often occurs in the presence of outlier(s). The WLSE incorporates an additional weight factor into the function above to quantify the importance of each observation in the data when estimating model parameters. The WLSE is (often) less sensitive to outliers when compared to the usual LSE[1]. The weighted least squares estimates can be obtained by maximizing

$$-\sum_{i=1}^{n} w_{(i)} \left[F(x_{(i)}) - \frac{i}{n+1} \right]^2, \tag{33}$$

where $w_{(i)} = \frac{(n+1)^2(n+2)}{i(n-i+1)}$ is the value of weight factor of the ith observation for the data in increasing order, [28, p. 181]. Similar to the least squares estimates, the weighted least squares estimates for the EGRL distribution are obtained by maximizing

$$-\sum_{i=1}^{n} w_{(i)} \left[\left(1 - \left[1 - G(x_{(i)})\right]^{\alpha}\right)^{\beta} - \frac{i}{n+1} \right]^2. \tag{34}$$

Another popular estimation method that can easily be applied to estimate model parameters for the EGRL distribution is the method of Cramer-Von-Mises estimation (CVME), which will be detailed in the next subsection.

3.3. **The method of Cramer-von-Mises estimation.** The estimates of model parameters using the Cramer-von-Mises estimation (CVME; [20]) is obtained by maximizing another function of the cdf of EGRL distribution which is given by

$$-\frac{1}{12n} - \sum_{i=1}^{n} \left[F(x_{(i)}) - \frac{2i-1}{2n} \right]^2. \tag{35}$$

Similar to equations (34) and (36), this function for the EGRL distribution can be defined as

$$-\frac{1}{12n} - \sum_{i=1}^{n} \left[\left(1 - \left[1 - G(x_{(i)})\right]^{\alpha}\right)^{\beta} - \frac{2i-1}{2n} \right]^2. \tag{36}$$

The gradients and analytical expressions of Hessian matrices with respect to the maximized functions using LSE, WLSE, and CVME methods are not presented here, but, would be made available upon request.

4. Simulation Studies

This section presents two simulation studies, first of which aims to investigate the performances of MLE, LSE, WLSE, and CVME methods for the EGRL distribution with respect to the bias, precision, and accuracy measures given in Walther and Moore ([34]). The second simulation study is however set up to illustrate

[1]The estimates obtained by the WLSE are not always resistant to outliers. For more details on the situations in which these estimates are sensitive to outliers, see Sohn et al. ([32]).

the potentiality of the new EGRL distribution in comparison to the some other lifetime distributions listed in Table 2. In this simulation, we show that model

TABLE 2. Some selected lifetime distributions.

Distribution	Author(s)
Rayleigh	Rayleigh ([25])
Exponentiated generalized Normal (EGN)	Corderio et al. ([7])
Exponentiated generalized Gumbel (EGGu)	Corderio et al. ([7])
Exponentiated generalized Ramos-Louzada (EGRL)	(New)

fit indices should be used in conjunction with information criteria to detect the best distribution in a set of distributions when analyzing the data. For the selection of best fitting models, the Cramer-von Mises (C*; [9]), Watson (W*; [35]), Kuiper (K*; [17]) and Kolmogorov-Smirnov (KS*; [16,31]) goodness of fit statistics and the log likelihood (ℓ), Akaike information criterion (AIC; [1, 2]), Consistent Akaike information criterion (CAIC; [6]), Corrected Akaike information criterion (AICc; [14]), Bayesian information criterion (BIC; [29]), and Hannan-Quinn information criterion (HQIC; [12]) are used. The goodness of fit statistics are given by

$$
\begin{aligned}
C^* =& \frac{1}{12n} + \sum_{k=1}^{n}\Big[\frac{2k-1}{2n} - F(X_{(k)})\Big]^2,\\
\mathrm{W}^* =& \sqrt{C^* - n\Big(\big[\frac{1}{n}\sum_{k=1}^{n}F(X_{(k)})\big] - \frac{1}{2}\Big)^2},\\
\mathrm{K}^* =& \max\big[\frac{k}{n} - F(x_{(k)})\big] + \max\big[F(X_{(k)}) - \frac{k-1}{n}\big],\\
\mathrm{KS}^* =& \max\Big[F(X_{(k)}) - \frac{k-1}{n}, \frac{k}{n} - F(X_{(k)})\Big],
\end{aligned} \tag{37}
$$

where n is the sample size and $F(x)$ is the cdf of the distribution under consideration for which the values of random variable X are in increasing order, namely, $x_{(1)} < x_{(2)} < ... < x_{(n)}$. The small values of these information criteria and test statistics above imply better model fits. Similarly, the information criteria are given by

$$
\begin{aligned}
AIC =& -2\ell + 2p,\\
\mathrm{AICc} =& -2\ell + \frac{2pn}{n-p-1},\\
\mathrm{CAIC} =& -2\ell + p\big[\log(n) + 1\big],
\end{aligned}
$$

$$\begin{aligned} \mathrm{BIC} &= -\,2\ell + p\log(n), \\ \mathrm{HQIC} &= -\,2\ell + 2p\log\big[\log(n)\big], \end{aligned} \tag{38}$$

where n is the sample size and p is the number of parameters in the model.

The simulation studies for two sets of population values of parameters for the EGRL distribution comprise the following steps.

(1) (a) *For the first simulation*: Set $\alpha = 1$, $\beta = 1$, and $\theta = 0.3$ as the population values of parameters for the EGRL distribution.
  (b) *For the second simulation*: Set $\alpha = 1.2$, $\beta = 1.3$, and $\theta = 0.3$ as the population values of parameters for the EGRL distribution.

(2) Set the sample size as $N = 20$, 100, or 500.

(3) Generate the values of EGRL distribution based on the population values in Step 1 and the sample size in Step 2. The data generation from the EGRL distribution is performed by using the automatic nonuniform random variate generation process presented in Hörmann et al. ([13]). This procedure can easily be implemented using `tdr.new` and `ur` subroutines of `Runuran` package in `R` statistical software.

(4) Obtain the estimates of model parameters using MLE, LSE, WLSE, and CVME methods for the EGRL distribution in the first simulation and for the distributions in Table 2 in the second simulation.

(5) Perform Steps 3-4 for $S = 1000$ times.

(6) (a) *For the first simulation*: For the EGRL distribution and each estimation method, calculate the bias, precision, and accuracy measures given in Walther and Moore ([34]).
  (b) *For the second simulation*: For each distribution and estimation method, calculate the log likelihood value, the values of goodness of fit statistics, and information criteria in equations (31), (39) and (40), respectively.

Notably, the measures in Step 6 (a) are obtained for each parameter of EGRL distribution in $S = 1000$ simulations. For example, the bias, precision, and accuracy measures for parameter α are given by

$$\mathrm{Bias}(\alpha) = \frac{1}{S}\sum_{s=1}^{S}(\hat{\alpha}_s - \alpha), \tag{39}$$

$$\mathrm{Precision}(\alpha) = \frac{1}{S}\sum_{s=1}^{S}(\hat{\alpha}_s - \bar{\alpha})^2, \tag{40}$$

$$\mathrm{Accuracy}(\alpha) = \frac{1}{S}\sum_{s=1}^{S}(\hat{\alpha}_s - \alpha)^2, \tag{41}$$

where $\alpha = 1$ is the population value of α in Step 1 (a), $\hat{\alpha}_s$ is the estimate of parameter α in the sth simulation, and $\bar{\alpha} = \frac{1}{S}\sum_{s=1}^{S}\hat{\alpha}_s$ for $s = 1, 2, ..., 1000$. Analogous calculations are performed for parameters β and θ. These values are displayed in

Table 3. Similarly, the values obtained in Step 6 (b) are presented in Tables 4, 5, 6, and 7 in each of which one of the estimation methods concerned are displayed in turn.

In the first and second simulation studies, parameter estimation over the data sets was performed by NM and BFGS algorithms, respectively. A data set was not accepted for inclusion in $S = 1000$ simulation trials if at least one of the following conditions occured. (1) The initial of parameter θ was not in the range of 0 and 0.5 or sample mean is smaller than 4 in line with $\theta_{\text{init}} = \frac{2}{\bar{x}+\sqrt{\bar{x}^2-4\bar{x}}}$ (see page 11). (2) The estimates of parameters were obtained outside the parameter space. For example, when the estimate of parameter $\alpha > 0$ for the EGN distribution is obtained as $\hat{\alpha} < 0$. (3) When the convergence criterion was not obtained for any of the distributions under consideration. (4) When the log likelihood value for any distribution in the set was obtained as minus infinity. Note that this last condition only applies to the second simulation study. If at least one of the conditions above occurs in the simulation, a different data was generated for the corresponding simulation trial.

Table 3 shows the values of bias, precision, and accuracy measures for each parameter of the EGRL distribution obtained from $S = 1000$ random datasets using MLE, LSE, WLSE, and CVME methods with NM algorithm. A small value in the table represents a small bias, a high precision, or a high accuracy measure. This table displays that increasing the sample size eventually reduces the bias and increases the precision and accuracy for parameters α, β, and θ of the EGRL distribution using each estimation method. The performance of each estimation method improves as the sample size increases. It is concluded that MLE outperforms other estimation methods in terms of bias, precision, and accuracy measures.

Tables 4, 5, 6, and 7 show the average values of the (minus) log likelihood, goodness of fit statistics, and information criteria for each distribution under evaluation using MLE, LSE, WLSE, and CVME methods with BFGS algorithm, respectively. Based on these tables, one-parameter Rayleigh distribution does not provide enough flexibility in modeling the data. Because model fit statistics and (minus) log likelihood values for this distribution are larger than other distributions. It seems that the EGRL distribution often has smaller, and thus, better model fit statistics and (minus) log likelihood values when compared to other distributions. However, note that, these goodness of fit statistics are biased themselves, since they do not take the model complexity into account when choosing the best distribution in a set of distributions. The information criteria like the AIC and BIC reduce this bias by penalizing model complexity (i.e., penalizing the models containing unnecessarily more parameters). For example, when estimating model parameters using MLE for $n = 20$ in Table 4, the best distribution in the set according to the values of model fit statistics is the EGGu distribution, while it is the second best distribution after the EGRL distribution based on the values of all the information criteria under consideration. Sample size plays a crucial role for information criteria when detecting the best distribution in a set. Because small samples tend to support

TABLE 3. Bias, precision and accuracy measures for the parameters of EGRL distribution

	MLE				LSE			
	$\hat{\alpha}$	Bias(α)	Precision(α)	Accuracy(α)	$\hat{\alpha}$	Bias(α)	Precision(α)	Accuracy(α)
$n = 20$	0.968	-0.032	0.300	0.301	0.685	-0.315	0.045	0.144
$n = 100$	0.991	-0.009	0.267	0.267	0.838	-0.162	0.048	0.075
$n = 500$	1.011	0.011	0.102	0.103	0.952	-0.048	0.059	0.061
	$\hat{\beta}$	Bias(β)	Precision(β)	Accuracy(β)	$\hat{\beta}$	Bias(β)	Precision(β)	Accuracy(β)
$n = 20$	1.027	0.027	0.081	0.081	0.853	-0.147	0.060	0.082
$n = 100$	0.958	-0.042	0.030	0.031	0.892	-0.108	0.019	0.031
$n = 500$	0.966	-0.034	0.014	0.015	0.941	-0.059	0.012	0.016
	$\hat{\theta}$	Bias(θ)	Precision(θ)	Accuracy(θ)	$\hat{\theta}$	Bias(θ)	Precision(θ)	Accuracy(θ)
$n = 20$	0.336	0.036	0.006	0.007	0.368	0.068	0.005	0.009
$n = 100$	0.339	0.039	0.007	0.009	0.360	0.060	0.005	0.009
$n = 500$	0.317	0.017	0.006	0.007	0.326	0.026	0.006	0.006
	WLSE				CVME			
	$\hat{\alpha}$	Bias(α)	Precision(α)	Accuracy(α)	$\hat{\alpha}$	Bias(α)	Precision(α)	Accuracy(α)
$n = 20$	0.689	-0.311	0.042	0.139	0.788	-0.212	0.053	0.098
$n = 100$	0.843	-0.157	0.046	0.071	0.868	-0.132	0.049	0.067
$n = 500$	0.953	-0.047	0.055	0.057	0.959	-0.041	0.058	0.059
	$\hat{\beta}$	Bias(β)	Precision(β)	Accuracy(β)	$\hat{\beta}$	Bias(β)	Precision(β)	Accuracy(β)
$n = 20$	0.849	-0.151	0.054	0.077	0.977	-0.023	0.088	0.089
$n = 100$	0.895	-0.105	0.019	0.030	0.921	-0.079	0.021	0.027
$n = 500$	0.946	-0.054	0.011	0.014	0.948	-0.052	0.012	0.015
	$\hat{\theta}$	Bias(θ)	Precision(θ)	Accuracy(θ)	$\hat{\theta}$	Bias(θ)	Precision(θ)	Accuracy(θ)
$n = 20$	0.378	0.078	0.004	0.010	0.362	0.062	0.005	0.008
$n = 100$	0.364	0.064	0.005	0.009	0.356	0.056	0.005	0.008
$n = 500$	0.327	0.027	0.005	0.006	0.325	0.025	0.006	0.006

more parsimonious (stated otherwise simple) models, while large samples tend to support more complex models. The performance of EGRL distribution increase better than that of other distributions in the set as the sample size increases. The EGRL distribution in these tables is associated with the smallest (average) values of information criteria for $n = 500$, regardless of the type of information criterion or estimation method. Moreover, the EGRL distribution performs better than other exponentiated generalized distributions, namely, the EGN and EGGu distributions, in all cases where the sample size is $n = 100$ or $n = 500$.

Table 4. The values of the goodness of fit statistics, (minus) log likelihood, and information criteria for each distribution under evaluation using MLE method.

Sample size (n)	Models	$\overline{\text{C}^*}$	$\overline{\text{W}^*}$	$\overline{\text{K}^*}$	$\overline{\text{KS}^*}$	-$\bar{\ell}$	$\overline{\text{AIC}}$	$\overline{\text{AICc}}$	$\overline{\text{CAIC}}$	$\overline{\text{BIC}}$	$\overline{\text{HQIC}}$
20	Rayleigh	0.44	0.43	0.37	0.28	57.00	115.99	116.21	117.99	116.99	116.18
	EGRL	0.06	0.23	0.23	0.14	51.33	108.67	111.04	114.66	111.66	109.25
	EGN	0.12	0.30	0.28	0.17	54.69	117.38	119.12	125.36	121.36	118.16
	EGGu	0.06	0.22	0.23	0.13	51.57	111.14	113.81	119.13	115.13	111.92
100	Rayleigh	1.69	0.79	0.27	0.22	276.01	554.02	554.07	557.63	556.63	555.08
	EGRL	0.06	0.22	0.10	0.06	251.62	509.23	509.63	520.05	517.05	512.40
	EGN	0.38	0.52	0.20	0.12	268.78	545.57	545.84	559.99	555.99	549.78
	EGGu	0.07	0.25	0.11	0.07	253.58	515.16	515.59	529.58	525.58	519.38
500	Rayleigh	7.74	1.66	0.24	0.20	1369.52	2741.03	2741.04	2746.25	2745.25	2742.68
	EGRL	0.05	0.21	0.05	0.03	1255.06	2516.11	2516.19	2531.76	2528.76	2521.08
	EGN	1.33	0.97	0.16	0.09	1328.86	2665.72	2665.77	2686.58	2682.58	2672.33
	EGGu	0.15	0.35	0.07	0.04	1265.08	2538.17	2538.25	2559.03	2555.03	2544.78

Table 5. The values of the goodness of fit statistics, (minus) log likelihood, and information criteria for each distribution under evaluation using LSE method.

Sample size (n)	Models	$\overline{\text{C}^*}$	$\overline{\text{W}^*}$	$\overline{\text{K}^*}$	$\overline{\text{KS}^*}$	-$\bar{\ell}$	$\overline{\text{AIC}}$	$\overline{\text{AICc}}$	$\overline{\text{CAIC}}$	$\overline{\text{BIC}}$	$\overline{\text{HQIC}}$
20	Rayleigh	0.17	0.38	0.34	0.20	59.43	120.85	121.08	122.85	121.85	121.05
	EGRL	0.04	0.21	0.22	0.12	51.14	108.27	110.65	114.26	111.26	108.86
	EGN	0.06	0.23	0.24	0.13	60.90	129.81	131.54	137.79	133.79	130.59
	EGGu	0.04	0.19	0.20	0.11	52.29	112.58	115.25	120.57	116.57	113.36
100	Rayleigh	0.63	0.77	0.26	0.14	288.01	578.01	578.05	581.62	580.62	579.07
	EGRL	0.04	0.19	0.10	0.05	251.20	508.39	508.79	519.21	516.21	511.56
	EGN	0.14	0.35	0.16	0.10	285.02	578.05	578.32	592.47	588.47	582.26
	EGGu	0.04	0.18	0.09	0.05	256.61	521.23	521.65	535.65	531.65	525.44
500	Rayleigh	2.89	1.68	0.23	0.12	1437.51	2877.03	2877.03	2882.24	2881.24	2878.68
	EGRL	0.04	0.18	0.04	0.02	1255.46	2516.92	2516.99	2532.56	2529.56	2521.88
	EGN	0.54	0.70	0.13	0.08	1391.44	2790.87	2790.92	2811.73	2807.73	2797.49
	EGGu	0.04	0.20	0.05	0.03	1286.80	2581.61	2581.69	2602.46	2598.46	2588.22

5. Application

This data contain $N = 116$ observations representing a mean ozone in parts per billion at Rosevelt Island. These observations are obtained from the `airquality` dataset in `datasets` package of `R` statistical software (version 4.2.2). Table 8 shows

TABLE 6. The values of the goodness of fit statistics, (minus) log likelihood, and information criteria for each distribution under evaluation using WLSE method.

Sample size (n)	Models	$\overline{C^*}$	$\overline{W^*}$	$\overline{K^*}$	$\overline{KS^*}$	$-\bar{\ell}$	$\overline{AIC}$	$\overline{AICc}$	$\overline{CAIC}$	$\overline{BIC}$	$\overline{HQIC}$
20	Rayleigh	0.17	0.38	0.33	0.20	59.04	120.09	120.31	122.08	121.08	120.28
	EGRL	0.05	0.21	0.22	0.12	50.92	107.83	110.21	113.82	110.82	108.42
	EGN	0.06	0.24	0.24	0.13	60.61	129.22	130.96	137.21	133.21	130.00
	EGGu	0.04	0.19	0.20	0.11	52.24	112.48	115.15	120.47	116.47	113.26
100	Rayleigh	0.63	0.76	0.26	0.15	286.51	575.02	575.06	578.62	577.62	576.07
	EGRL	0.04	0.20	0.10	0.05	250.92	507.83	508.23	518.65	515.65	510.99
	EGN	0.16	0.38	0.16	0.09	372.13	752.26	752.53	766.68	762.68	756.48
	EGGu	0.04	0.18	0.09	0.05	256.45	520.91	521.33	535.33	531.33	525.12
500	Rayleigh	2.91	1.66	0.23	0.12	1428.25	2858.50	2858.51	2863.71	2862.71	2860.15
	EGRL	0.04	0.19	0.04	0.02	1255.35	2516.69	2516.77	2532.34	2529.34	2521.65
	EGN	0.56	0.72	0.15	0.10	3498.90	7005.81	7005.86	7026.67	7022.67	7012.42
	EGGu	0.04	0.20	0.05	0.03	1286.84	2581.68	2581.76	2602.54	2598.54	2588.29

TABLE 7. The values of the goodness of fit statistics, (minus) log likelihood, and information criteria for each distribution under evaluation using CVME method.

Sample size (n)	Models	$\overline{C^*}$	$\overline{W^*}$	$\overline{K^*}$	$\overline{KS^*}$	$-\bar{\ell}$	$\overline{AIC}$	$\overline{AICc}$	$\overline{CAIC}$	$\overline{BIC}$	$\overline{HQIC}$
20	Rayleigh	0.18	0.39	0.34	0.20	59.83	121.66	121.88	123.65	122.65	121.85
	EGRL	0.04	0.20	0.21	0.11	51.28	108.55	110.93	114.54	111.54	109.14
	EGN	0.05	0.22	0.23	0.13	62.26	132.52	134.25	140.50	136.50	133.30
	EGGu	0.03	0.18	0.19	0.10	52.73	113.45	116.12	121.44	117.44	114.23
100	Rayleigh	0.63	0.77	0.26	0.14	288.11	578.21	578.25	581.82	580.82	579.27
	EGRL	0.04	0.19	0.09	0.05	251.28	508.57	508.96	519.38	516.38	511.73
	EGN	0.14	0.35	0.16	0.09	283.67	575.33	575.61	589.75	585.75	579.55
	EGGu	0.04	0.18	0.09	0.05	257.24	522.47	522.90	536.90	532.90	526.69
500	Rayleigh	2.89	1.68	0.23	0.12	1437.64	2877.29	2877.29	2882.50	2881.50	2878.94
	EGRL	0.04	0.18	0.04	0.02	1255.44	2516.88	2516.95	2532.52	2529.52	2521.84
	EGN	0.54	0.70	0.13	0.08	1391.87	2791.74	2791.79	2812.60	2808.60	2798.36
	EGGu	0.04	0.20	0.05	0.02	1286.81	2581.61	2581.69	2602.47	2598.47	2588.23

the data and its descriptives. This dataset is heavily right skewed. The Q-Q plot in Figure 3 and Shapiro-Wilk normality test results ($W = 0.879$, $p < 0.001$) show that the dataset is not normally distributed. The boxplot in Figure 3 displays that the dataset contains outliers. Table 9 shows the estimates of model parameters for the Ozone data using each of the estimation methods. We provide the `R` code on how to obtain the estimates of model parameters using MLE in Appendix. The `R` code for other estimation methods and distributions are not presented in Appendix, but, would be made available upon request.

TABLE 8. The Ozone data and its descriptives.

Data:	41,	36,	12,	18,	28,	23,	19,	8,	7,	16,	11,	14,	18,	14,	34,	6,	30,
	11,	1,	11,	4,	32,	23,	45,	115,	37,	29,	71,	39,	23,	21,	37,	20,	12,
	13,	135,	49,	32,	64,	40,	77,	97,	97,	85,	10,	27,	7,	48,	35,	61,	79,
	63,	16,	80,	108,	20,	52,	82,	50,	64,	59,	39,	9,	16,	78,	35,	66,	122,
	89,	110,	44,	28,	65,	22,	59,	23,	31,	44,	21,	9,	45,	168,	73,	76,	118,
	84,	85,	96,	78,	73,	91,	47,	32,	20,	23,	21,	24,	44,	21,	28,	9,	13,
	46,	18,	13,	24,	16,	13,	23,	36,	7,	14,	30,	14,	18,	20			

Mean	Std. deviation	Median	Trimmed mean	Median abs. deviation	Min	Max	Range	Skewness	Kurtosis	Std. error
42.13	32.99	31.50	37.80	25.95	1.00	168.00	167.00	1.21	1.11	3.06

TABLE 9. The estimates of model parameters using MLE, LSE, WLSE, and CVME for the Ozone data.

	MLE				LSE			
Models	α	β	μ	θ	α	β	μ	θ
Rayleigh	37.774	–	–	–	28.295	–	–	–
EGRL	1.423	1.795	–	0.024	1.061	1.514	–	0.030
EGN	1.542	123.522	-114.284	84.077	3.237	115.191	-56.913	111.820
EGGu	0.182	0.602	16.227	7.177	0.115	0.648	12.322	5.188
	WLSE				CVM			
Models	α	β	μ	θ	α	β	μ	θ
Rayleigh	28.735	–	–	–	28.288	–	–	–
EGRL	2.999	1.654	–	0.011	1.893	1.551	–	0.017
EGN	0.452	218.532	-179.939	47.622	3.306	109.547	-51.712	110.068
EGGu	0.131	0.761	10.899	5.250	0.114	0.661	12.198	5.031

Figure 4 shows the pdfs, cdfs and survival and hazard rate functions for each distribution using MLE. This figure shows that the EGRL and EGGu distributions fit the data better than the Rayleigh and EGN distributions. Table 10 shows that the distribution of the observed Ozone data does not deviate significantly from the EGRL and EGGu distributions, but the distribution of the data differs from the Rayleigh and EGN distributions. This can be tested by the values of Kolmogorov-Smirnov (KS*) test statistics. For doing this, the critical value for the KS test is determined for $\alpha = 0.05$, that is, $KS_t = \frac{1.36}{\sqrt{n}} = \frac{1.36}{\sqrt{116}} = 0.126$. Therefore, for example, $KS^* = 0.085 < KS_t = 0.126$ and $KS^* = 0.065 < KS_t = 0.126$ indicate that the distribution of the Ozone data is not significantly different from the EGRL

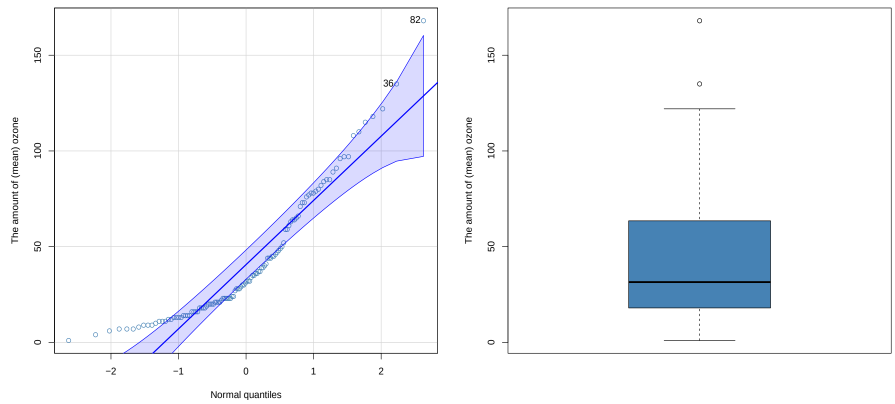

FIGURE 3. The Q-Q plot and box plot for the Ozone data.

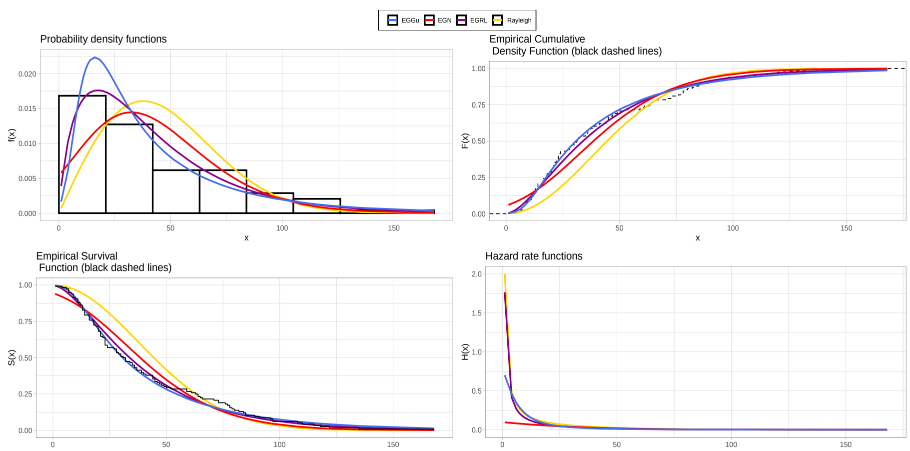

FIGURE 4. The pdf, cdf, survival and hazard rate functions for the Ozone data.

and EGGu distributions, respectively, when parameter estimation is performed by MLE. However, $KS^* = 0.248 > KS_t = 0.126$ and $KS^* = 0.214 > KS_t = 0.126$ mean that the distribution of the data significantly different from the Rayleigh and EGN distributions, respectively, when parameter estimation is performed by MLE. Table 10 also shows that the EGGu distribution provides the smallest goodness of fit statistics, regardless of the method used for parameter estimation. However, as noted in the introduction, these goodness of fit statistisc are biased as they do not take the model complexity into account. Information criteria reduce this bias by considering both the fit and complexity of the model being evaluated. In Table 10, the values of information criteria indicate that the EGRL distribution has a better balance between the model fit and complexity when compared to Rayleigh, EGN, and EGGu distributions. Thus, we conclude that the EGRL distribution can be considered as an alternative distribution in the exponential generalized family of distributions when analyzing positively skewed data using MLE, LSE, WLSE, and CVME for parameter estimation.

TABLE 10. The values of the goodness of fit statistics, (minus) log likelihood, and information criteria for each distribution under evaluation using MLE, LSE, WLSE, and CVME for the Ozone data.

						MLE				
Models	C^*	W^*	K^*	KS^*	$-\ell$	AIC	AICc	CAIC	BIC	HQIC
Rayleigh	2.17	1.03	0.31	0.25	561.99	1125.97	1126.01	1129.73	1128.73	1127.09
EGRL	0.12	0.33	0.14	0.09	541.40	1088.79	1089.01	1100.05	1097.05	1092.15
EGN	0.51	0.66	0.21	0.14	556.53	1121.06	1121.42	1136.08	1132.08	1125.53
EGGu	0.06	0.24	0.12	0.07	540.46	1088.91	1089.27	1103.93	1099.93	1093.39
						LSE				
Models	C^*	W^*	K^*	KS^*	$-\ell$	AIC	AICc	CAIC	BIC	HQIC
Rayleigh	0.87	0.93	0.31	0.18	585.69	1173.38	1173.41	1177.13	1176.13	1174.50
EGRL	0.08	0.28	0.12	0.06	542.15	1090.30	1090.52	1101.56	1098.56	1093.65
EGN	0.32	0.55	0.20	0.11	558.28	1124.56	1124.92	1139.57	1135.57	1129.03
EGGu	0.03	0.18	0.08	0.04	540.99	1089.98	1090.34	1105.00	1101.00	1094.45
						WLSE				
Models	C^*	W^*	K^*	KS^*	$-\ell$	AIC	AICc	CAIC	BIC	HQIC
Rayleigh	0.87	0.94	0.31	0.17	582.99	1167.98	1168.01	1171.73	1170.73	1169.10
EGRL	0.09	0.30	0.13	0.07	541.58	1089.15	1089.37	1100.41	1097.41	1092.51
EGN	0.39	0.62	0.22	0.12	556.00	1120.00	1120.35	1135.01	1131.01	1124.46
EGGu	0.03	0.18	0.10	0.05	540.28	1088.56	1088.92	1103.57	1099.57	1093.03
						CVME				
Models	C^*	W^*	K^*	KS^*	$-\ell$	AIC	AICc	CAIC	BIC	HQIC
Rayleigh	0.87	0.93	0.31	0.18	585.74	1173.48	1173.51	1177.23	1176.23	1174.59
EGRL	0.08	0.28	0.12	0.06	541.96	1089.92	1090.14	1101.18	1098.18	1093.28
EGN	0.32	0.55	0.20	0.11	558.29	1124.59	1124.95	1139.60	1135.60	1129.06
EGGu	0.03	0.17	0.08	0.04	540.98	1089.96	1090.32	1104.98	1100.98	1094.43

Goodness of fit statistics and information criteria are originally created to compare the performance of models, but not to compare the performance of estimation methods. Therefore, we do *not* recommend using the results in Table 10 to compare the performance of the methods in estimating the parameters of the EGRL distribution. For doing this, we used a bootstrap estimate of bias presented in Efron and Tibshirani ([10]).

Let $\eta = (\alpha, \beta, \theta)$ be the vector containing the parameters of EGRL distribution. Then, the bootstrap estimate of bias for each estimation method is calculated as follows:

(1) Create $B = 1000$ bootstrap samples by resampling with replacement from the original data.
(2) Obtain the estimate of parameter vector η for each of the bootstrap samples.
(3) Obtain the overall bootstrap estimate of parameter vector η, that is, η^*, by averaging the estimates among the bootstrap samples. The standard errors of parameter estimates (i.e., SE_B^*) are obtained by taking the square root of diagonals of the covariance matrix for the estimates in the bootstrap samples.
(4) Calculate the bootstrap estimate of bias for parameter vector η, that is, $\mathrm{Bias}_\mathrm{B} = |\eta^* - \hat{\eta}|$, where $\hat{\eta}$ is the vector of the usual estimates obtained for the original data using MLE, LSE, WLSE, or CVME.

Table 11 shows the performance evaluation of each estimation method in estimating the parameters of the EGRL distribution for the Ozone data. Efron and Tibshirani ([10]) state that the bias can be ignored if $\frac{Bias_B}{SE_B^*} \leq 0.25$. Therefore, based on the results in Table 11, CVME outperforms other estimation methods for this particular example as it has the ratios smaller than 0.25 when estimating parameters α, β, and θ. In line with Efron and Tibshirani ([10]), the bias-adjusted estimates (BAEs) are also provided for each estimation method using $2\hat{\eta} - \eta^*$. However, caution should be taken when using the bias-adjusted estimates in place of the usual estimates, as biases are more difficult to estimate than standard errors and correcting bias may produce higher standard errors [10, p. 138]. While the bias-adjusted estimates are reasonably close to the usual estimates for MLE and CVME, these estimates are not close to the usual estimates for LSE and WLSE. The WLSE even produces a negative bias-adjusted estimate for parameter θ.

In summary, it is concluded that CVME performs better than other estimation methods in analyzing the Ozone data based on nonparametric bootstrapping bias assessment. In this sense, MLE also provides a reasonable set of parameter estimates. However, LSE and WLSE do not perform well when compared to MLE and CVME for analyzing the Ozone data using the EGRL distribution.

6. Discussion

In this study, we introduced a new distribution called the exponentiated generalized Ramos-Louzada distribution involving three parameters. We used four estimation methods (i.e., the MLE, LSE, WLSE, and CVME) for estimation. We assess the performance of these methods for the EGRL distribution by means of

Table 11. Performance evaluation of the estimation methods for the Ozone data in terms of the bias measure using nonparametric bootstrapping.

Method	α	β	θ
MLE ($\hat{\eta}$)	1.423	1.795	0.024
NB (η^*)	1.016	1.830	0.046
SE_B^*	0.411	0.231	0.035
$\mathrm{Bias}_\mathrm{B} = \lvert\eta^* - \hat{\eta}\rvert$	0.407	0.035	0.022
Ratio $= \frac{\mathrm{Bias}_\mathrm{B}}{\mathrm{SE}_\mathrm{B}^*}$	0.990	0.152	0.629
BAE $= 2\hat{\eta} - \eta^*$	1.830	1.760	0.002
LSE ($\hat{\eta}$)	1.061	1.514	0.030
NB (η^*)	1.783	1.564	0.020
SE_B^*	0.431	0.225	0.012
$\mathrm{Bias}_\mathrm{B} = \lvert\eta^* - \hat{\eta}\rvert$	0.722	0.050	0.010
Ratio $= \frac{\mathrm{Bias}_\mathrm{B}}{\mathrm{SE}_\mathrm{B}^*}$	1.675	0.222	0.833
BAE $= 2\hat{\eta} - \eta^*$	0.339	1.464	0.040
WLSE ($\hat{\eta}$)	2.999	1.654	0.011
NB (η^*)	2.060	1.735	0.023
SE_B^*	0.836	0.217	0.023
$\mathrm{Bias}_\mathrm{B} = \lvert\eta^* - \hat{\eta}\rvert$	0.939	0.081	0.012
Ratio $= \frac{\mathrm{Bias}_\mathrm{B}}{\mathrm{SE}_\mathrm{B}^*}$	1.123	0.373	0.522
BAE $= 2\hat{\eta} - \eta^*$	3.938	1.573	-0.001
CVME ($\hat{\eta}$)	1.893	1.551	0.017
NB (η^*)	1.792	1.602	0.020
SE_B^*	0.439	0.233	0.013
$\mathrm{Bias}_\mathrm{B} = \lvert\eta^* - \hat{\eta}\rvert$	0.101	0.051	0.003
Ratio $= \frac{\mathrm{Bias}_\mathrm{B}}{\mathrm{SE}_\mathrm{B}^*}$	0.230	0.219	0.231
BAE $= 2\hat{\eta} - \eta^*$	1.994	1.500	0.014

using bias, precision, and accuracy measures, the goodness of fit statistics, and information criteria. To attain this objective, we first generate the datasets from the EGRL distribution in two simulations with varying values of sample size. Then, in the first simulation, we evaluate the performance of each estimation method for the EGRL distribution by means of using bias, precision, and accuracy measures. We obtain smaller bias and better precision and accuracy measures for each parameter of EGRL distribution as the sample size increases. It is concluded that MLE outperforms other estimation methods as the sample size increases. Second simulation study is conducted to evaluate the performance of a set of distributions for each estimation method separately. It is concluded that the performance of

EGRL distribution increase better than that of other distributions as the sample size increases when the data in fact follow the EGRL distribution.

The EGRL distribution is a flexible distribution that can be used to improve the model fit when compared to other exponentiated generalized distributions such as EGN and EGGu distributions. However, caution should be taken when using this distribution to analyze datasets in some certain circumstances. We are compelled to highlight two main limitations of the EGRL distribution when using it in conjunction with the estimation methods and information criteria presented in this paper. First, the performance of the EGRL distribution on modeling the data depends on the method utilized for estimation. For example, the methods presented in this paper might produce biased parameter estimates and their standard errors in the case of the data contain many missing values and/or outliers. In such cases, a different estimation method dealing with missing values and/or outliers better should be preferred over these estimation methods. Second, the AIC is prone to overfitting, that is, falsely choosing more complicated distributions containing more parameters over the simpler (stated otherwise more parsimonious) distributions for small samples. We do not suggest the use of the EGRL distribution for small samples, since it contains relatively more parameters when compared to the usual RL distribution. In the same sense, the AIC should not be used as a decision criterion when the set of distributions contains the EGRL distribution for small samples.

Author Contribution Statements Writing, programming, and analysis was performed by Yasin Altınışık. All authors read, commented, and approved the final article.

Declaration of Competing Interests The authors declare that they have no competing interest.

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Appendix

The R (version 4.2.2) code used to estimate the parameters of the EGRL distribution using MLE is given below.

```
library(maxLik)
set.seed(111)
# The Ozone data
xi <- c(41, 36, 12, 18, 28, 23, 19, 8, 7, 16, 11, 14, 18, 14, 34,
6, 30, 11, 1, 11, 4, 32, 23, 45, 115, 37, 29, 71, 39, 23, 21, 37,
20, 12, 13, 135, 49, 32, 64, 40, 77, 97, 97, 85, 10, 27, 7, 48, 35,
61, 79, 63, 16, 80, 108, 20, 52, 82, 50, 64, 59, 39, 9, 16, 78, 35,
66, 122, 89, 110, 44, 28, 65, 22, 59, 23, 31, 44, 21, 9, 45, 168, 73,
76, 118, 84, 85, 96, 78, 73, 91, 47, 32, 20, 23, 21, 24, 44, 21, 28,
9, 13, 46, 18, 13, 24, 16, 13, 23, 36, 7, 14, 30, 14, 18, 20)
# Sample size
n <- length(xi)
# Determining the initials for the parameters, respectively.
alphainit <- 1
betainit <- 1
thetainit<-2/(mean(xi)+sqrt(mean(xi)^2-4*mean(xi)))
# Maximizing the log likelihood function.
logLik <- function(param) {
alpha <- param[1]
beta <- param[2]
```

```
theta <- param[3]
# The four lines below are used to ensure that the estimates updated
# in the BFGS algorithm are in line with the parameter spaces.
if (alpha < 0) {alpha <- 0.0001}
if (beta < 0) {beta <- 0.0001}
if (theta < 0) {theta <- 0.0001}
if (theta > 0.5) {theta <- 0.5}
gx <- (((1+theta^2*xi-2*theta)*(theta))/(1-theta))*(exp(-theta*xi))
Gx <- 1-((1+theta^2*xi-theta)/(1-theta))*(exp(-theta*xi))
ll <- n*log(alpha)+n*log(beta)+sum(log(gx))+(alpha-1)*(sum(log(1-Gx)))+
(beta-1)*sum(log(1-((1-Gx)^(alpha))))
}
# Obtaining the results of the BFGS algorithm. Here,
# control = list(iterlim = 100000) is used to ensure successfull
# convergence of the BFGS algorithm.
model <- maxLik(logLik, start = c(alphainit, betainit, thetainit),
method = "BFGS", control = list(iterlim = 100000))
# Displaying the results
summary(model)
--------------------------------------------
Maximum Likelihood estimation
BFGS maximization, 44 iterations
Return code 0: successful convergence
Log-Likelihood: -541.3966
3  free parameters
Estimates:
     Estimate Std. error t value  Pr(> t)
[1,]  1.42313    2.40164   0.593    0.553
[2,]  1.79495    0.24685   7.271 3.56e-13 ***
[3,]  0.02424    0.04196   0.578    0.563
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
--------------------------------------------
```