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Rings and Modules Whose Socles are Relative Ejective

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Abstract

Lifting homomorphism from modules to modules or even from certain submodule to the modules have been important both in ring and module theory. In this note we study rings and modules whose socles are relative ejective. Moreover we reduce our consideration to rings and modules with injective socles which provides the dual notion to $PS-$modules.

Keywords: Socle, $M-$injective module, radical of a module

1. INTRODUCTION

Throughout this paper all rings are associative with identity and all modules are unital right modules. Let R be a ring and let M be an R–module. Then the *radical* of M is defined by the intersection of all maximal submodules of M or M if M has no maximal submodule and denoted $RadM$. Recall that for a right R–module M the *singular submodule* is defined by

$Z(M) = \{m \in M: mE = 0$ for some essential right ideal E of $R\}$

and a module M is called *nonsingular* provided that $Z(M) = 0$ (see [2]). Note that W.K. Nicholson and J.F. Watters called a module M a *PS–module* if every simple submodule of M is projective, equivalently $SocM$ is projective (see [4]). To this end it is natural to think of rings with injective radical dual to PS–rings. In this case it is easy to see that the Jacobson radical of R is zero. Recently relative ejectivity was defined (see [1] and [8]). Let N, M be R–modules. Then N is called *M–ejective* if for each submodule K of M and each homomorphism $\varphi: K \to N$ there exist a homomorphism $\theta: M \to N$ and an essential submodule X of K such that $\theta|_X = \varphi|_X$ i.e.; $\theta(x) = \varphi(x)$ for all $x \in X$. It is clear that every M–injective module is M–ejective. However, the converse is not true in general (see for example [1]).

In this paper we deal with modules and rings with M–ejective socles or injective socles. To this end, we obtain basic properties of EJS–modules and make sure that the class of EJS–modules is different from the class of weak CS–modules. For; unexplained terminology and notation we refer to [2], [3], [5]. So:

2. EJS–RINGS AND MODULES

Definition 1. Let M be an R–module. Then M is called an EJS (respectively INS)–*module* if $SocM$ is M–ejective (respectively injective). The ring R is said to be *right EJS (INS)–ring* whenever the right R–module R is an EJS *(INS)*–module.

Example 1. Let M be an R–module.

i. If $SocM$ is injective or M–injective then M is an EJS–module.
ii. If $SocM = 0$ then M is an EJS–module.

Observe that Example 1(ii) yields that in particular the rings of integers $\mathbb{Z}$ and the polynomial ring $R[x]$ over a ring R are EJS–rings.

The following Lemma is the part of Corollary 2.5 in [1] which motivates our work.

Lemma 1. [1, Corollary 2.5] If $SocM$ is essential in M, then N is M–ejective if and only if for each

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homomorphism $\varphi: SocM \to N$ there exists a homomorphism $\theta: M \to N$ such that $\theta|_{SocM} = \varphi$. In other words, the diagram

$$\begin{tikzcd} SocM \arrow[r, "i"] \arrow[d, "\varphi"'] & M \arrow[dl, dotted, "\theta"] \\ N & \end{tikzcd}$$

commutes where i denotes the inclusion mapping.

As the following example illustrates the condition $SocM$ is being essential in M is not superfluous in Lemma 1.

Example 2. Let p be any prime integer and let $M = (\mathbb{Z}/\mathbb{Z}p) \oplus \mathbb{Q}$ be the $\mathbb{Z}$–module. Then

i. $SocM$ is not M–ejective.
ii. Any $\varphi \in Hom_{\mathbb{Z}}(SocM, SocM)$ can be lifted to $\theta \in Hom_{\mathbb{Z}}(M, SocM)$.

Proof. First of all note that

$$SocM = \mathbb{Z}/\mathbb{Z}p \oplus 0 = N$$

is not essential in $M_{\mathbb{Z}}$.

i. Assume to the contrary and let $\pi: \mathbb{Z} \to N$ be the canonical epimorphism. Thus there exists a homomorphism $\theta: M \to N$ such that $\theta|_{\mathbb{Z}} = \pi$. In particular $\alpha = \theta|_{\mathbb{Q}}: \mathbb{Q} \to N$ lifts π i.e.; $\alpha|_{\mathbb{Z}} = \pi$.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "\pi"'] & \mathbb{Q} \arrow[dl, dotted, "\alpha"] \\ & \mathbb{Z}/\mathbb{Z}p & \end{tikzcd}$$

Now $\alpha(1/p) = x + \mathbb{Z}p$ for some $x \in \mathbb{Z}$. Thus $p\alpha(1/p) = \alpha(1) = \pi(1) = 1 + \mathbb{Z}p$. It follows that $px + \mathbb{Z}p = 1 + \mathbb{Z}p$ and hence $1 \equiv 0$ $(mod\ p)$, a contradiction. Hence $SocM$ is not M–ejective.

ii. Let $\varphi: N \to N$ be any homomorphism. If $\varphi = 0$ then we have done. It follows that $\varphi = i$. Define $\pi: M \to N$ by $\pi(x + \mathbb{Z}p, y) = x + \mathbb{Z}p$. It is clear that $\pi|_N = i$. □

Proposition 1. Assume $SocM$ is essential in M. Then M is an EJS–module if and only if M is semisimple.

Proof. (⇐) This part is clear.

(⇒) Suppose M is an EJS–module. Let $i: SocM \to SocM$ be the identity mapping. By Lemma 1, there exists a homomorphism $\theta: M \to SocM$ such that $\theta|_{SocM} = i$. Then $M = SocM + Ker\theta$. Let $x \in SocM \cap Ker\theta$. Thus

$$x = \theta(x) = 0.$$

Hence $M = SocM \oplus Ker\theta$. Since $SocM$ is essential in M, we obtain that

$$SocM = M$$

i.e.; M is semisimple. □

Our next objective is to give an example which illustrates the former result. Incidentally, recall that a module M is called *weak CS* if every semisimple submodule is essential in a direct summand of M, see for example [7] or related references there in.

Example 3. Let p be prime number and A be $\mathbb{Z}$–module $(\mathbb{Z}/\mathbb{Z}p) \oplus (\mathbb{Z}/\mathbb{Z}p^3)$. Then A is a weak CS–module with essential socle which is not EJS–module.

Proof. It is clear that $SocA \leq_e A_{\mathbb{Z}}$. Now let us show that A is a weak CS–module. Note that A has uniform dimension 2. Let S be a semisimple submodule of A. If S is not simple, $S \leq_e A$. Suppose that S is simple. Then $S = (a + \mathbb{Z}p, p^2b + \mathbb{Z}p^3)\mathbb{Z}$ for some integers a, b such that $0 \leq a, b \leq p - 1$. If $a = 0$, then $S \leq_e L = 0 \oplus (\mathbb{Z}/\mathbb{Z}p^3)$. If $a \neq 0$, then $A = S \oplus L$. Thus, in any case, S is essential in a direct summand of A. Thus $A_{\mathbb{Z}}$ is a weak CS–module. Since $A_{\mathbb{Z}}$ is not semisimple, it is not EJS–module by Proposition 1. □

One might expect that whether the EJS property implies weak CS condition or not? However there are several examples which eliminate this possibility. For example, let p be prime number and let $A = (\mathbb{Z}/\mathbb{Z}p) \oplus \mathbb{Z}$ be the $\mathbb{Z}$–module. Now, let us form the trivial extension of $\mathbb{Z}$ with A i.e.;

$$R = \begin{bmatrix} \mathbb{Z} & A \\ 0 & \mathbb{Z} \end{bmatrix} = \left\{ \begin{bmatrix} n & (\bar{x}, y) \\ 0 & n \end{bmatrix} : n \in \mathbb{Z}, (\bar{x}, y) \in A \right\}.$$

Then $Soc(R_R) = \begin{bmatrix} 0 & \mathbb{Z}/\mathbb{Z}p \oplus 0 \\ 0 & 0 \end{bmatrix}$ which is not essential in R_R. It follows that R_R is not a weak CS–module. On the other hand, it is straightforward to see that R_R is an EJS–module.

Lemma 2. Let A be an Abelian group (i.e.; $\mathbb{Z}$–module). Then

i. $RadA = \bigcap_{p \text{ prime}} pA$.
ii. If A is torsion then $RadA = 0$ if and only if A is semisimple.

Proof. i. It is easy to check.

ii. (⇐) Clear.

(⇒) Let $A = \bigoplus_{p \text{ prime}} A_p$ where A_p is a torsion p–group. Let q be any prime such that $q \neq p$. Let $x \in A_p$. Then $p^n x = 0$ for some $n \geq 1$. Also $1 = sq + tp^n$ for some $s, t \in \mathbb{Z}$. It follows that $x =$

$sqx + tp^n x = q(sx) \in qA_p$. Therefore $A_p = qA_p$ for all primes $q \neq p$. Thus

$$RadA_p = \left(\bigcap_{q \text{ prime}, \ q \neq p} qA_p\right) \cap (pA_p) = A_p \cap pA_p = 0.$$

It follows that $A_p \cong A_p/pA_p$ so A_p is semisimple and hence A is semisimple. □

Combining Lemma 2(ii) together with Proposition 1, we have the next result.

Theorem 1. Let A be a torsion Abelian group. Then the following statements are equivalent.

i. $RadA = 0$.
ii. A is semisimple.
iii. A is an EJS–module.

Proof. (i) ⇔ (ii) By Lemma 2(ii).

(ii) ⇔ (iii) By Proposition 1. □

Proposition 2. Let R be an EJS–ring. Then every projective simple right R–module is an EJS–module.

Proof. Suppose X is projective simple R–module. Then $X = xR$ for some $0 \neq x \in X$. Since $R/r(x) \cong X$ is projective simple where $r(x)$ is the right annihilator of x in R. Then $R = r(x) \oplus E$ for some $E \leq R$. Now

$$E \cong R/r(x)$$

is projective simple. Hence $E \leq SocR$. Then $SocR = E \oplus F$ for some right ideal F of R. Thus E is an EJS–module. Therefore

$$X \cong R/r(x) \cong E$$

is an EJS–module. □

Now we focus on the case in which that $SocM$ is an injective module. Recall that a module M is called an *INS–module* if $SocM$ is injective. Also a ring R is called right *INS–ring* if R_R is an INS–module. We continue with the following easy Lemma.

Lemma 3. The class of INS–modules is closed under direct products, submodules and essential extensions.

Proof. It is straightforward to check. □

Proposition 3. Let R be a ring. Then R is a right INS–ring if and only if R has a faithful right INS–module.

Proof. (⇒) Obvious.

(⇐) Suppose M is an INS–module. Then R_R embeds in $\prod M$. Thus $R \cong X \leq \prod M$. Since $SocR \cong SocX \leq Soc(\prod M)$, $SocR$ is injective.□

Theorem 2. Let R be a ring. Then R is an INS–ring if and only if the following conditions hold.

i. $SocR$ is finitely generated and projective.
ii. Every projective simple right R–module is injective.

Proof. Assume that (i) and (ii) hold. $SocR = U_1 \oplus U_2 \oplus \ldots \oplus U_n$ where U_i's are simple and projective. Thus $SocR$ is injective.

Assume that R is an INS–ring. By hypothesis, $R = SocR \oplus F$ for some right ideal F of R. Thus $SocR$ is cyclic and projective. Now Proposition 2 completes the proof. □

Theorem 3. Let R be a ring. Then $SocR = eR$ for some $e^2 = e \in R$ if and only if $R = S \oplus T$ where S is semiprime Artinian ring and T is a ring with zero right socle.

Proof. Suppose $SocR = eR$ where $e^2 = e \in R$. Thus $(1-e)Re = 0$. Hence

$$R \cong \begin{bmatrix} eRe & eR(1-e) \\ 0 & (1-e)R(1-e) \end{bmatrix} = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}.$$

Now $SocR = \begin{bmatrix} S & M \\ 0 & 0 \end{bmatrix}$. Since

$$\begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} \leq R \quad \text{and} \quad \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} \leq SocR,$$

$\begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} = fR$ for some $f^2 = f \in R$. It follows that $M = 0$. So [6] yields that $R \cong S \oplus T$ where S is semiprime Artinian and T has zero socle.

Conversely, let A be a right ideal of T and let $\varphi: A \to S$ be homomorphism. Now $\varphi(S) \leq S$ and $\varphi(A) = \varphi(A)S = \varphi(AS) \leq \varphi(A \cap S) = \varphi(0) = 0$. Hence $\varphi = 0$. Thus φ lifts to T. It follows that S is S–injective and T–injective. Therefore S is injective. □

Corollary 1. If R is an INS–ring then $R \cong S \oplus T$ where S is a semiprime Artinian ring and T is a ring with zero right socle.

Proof. Immediate by Theorem 3. □

Our next objective is to clarify when a nonsingular right R–module is an INS–module. To this end a nonsingular right R–module has a projective socle i.e.; it is a PS–module (see [4]).

Example 4. *Let* $M = [K \quad K]$ and $R = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$ where K is field. Then M is right nonsingular right R–module which is not an INS–module.

Proof. Suppose $SocM = \begin{bmatrix} 0 & K \end{bmatrix}$ is injective. Define a homomorphism $\varphi: \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & K \end{bmatrix}$ by $\varphi\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & x \end{bmatrix}$ for $x \in K$. Hence

$$\begin{aligned} \begin{bmatrix} 0 & 1 \end{bmatrix} = \varphi\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) &= \begin{bmatrix} 0 & y \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \end{bmatrix} \end{aligned}$$

for some $y \in K$. Which is a contradiction. It follows that $SocM$ is not injective. However it is clear that M is nonsingular. □

Corollary 2. Let R be commutative Noetherian ring and let M be a nonsingular R–module. Then M is an INS–module.

Proof. It is not difficult to see that any nonsingular simple R–module is injective. □

Observe that Theorem 3 leads us to think of generalized triangular matrix EJS (INS)–rings. For, let R be ring as in Example 4. It can be seen easily that R is not EJS (and hence not INS)–ring (see [8]). Incidentally, we should mention that there are trivial extensions which are not EJS–rings (see [8]). Furtermore, it will be an essential search to investigate relationships between the class of EJS–modules and generalizations of extending modules.

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