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SOME NEW RESULTS ON CONVERGENCE, STABILITY AND DATA DEPENDENCE IN $n-$NORMED SPACES

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Abstract. We introduce a new contractive condition and a new iterative method in $n-$ normed space setting. We employ both of these to study convergence, stability, and data dependence. The results presented here extend and improve some recent results announced in the existing literature.

1. Introduction

The theory of n-normed spaces has been introduced by Misiak [1] as a generalization of the theory of 2-normed spaces due to Gähler [2]. Since then, much effort has been devoted to the development of the theory of $n-$normed spaces. See, e.g. [3-5] and references therein. We recall some basic facts as follows.

Definition 1. *([1]) Let $n \in \mathbb{N}$ and X be a real vector space of dimension d, where $n \leq d$. A real-valued function $\|\cdot, \ldots, \cdot\| : X^n \to \mathbb{R}$ which satisfies the following conditions:*

(nN_1) $\|x_1, \ldots, x_n\| = 0$ iff $x_1, \ldots, x_n$ are linearly dependent,
(nN_2) $\|x_1, \ldots, x_n\|$ is invariant under any permutation,
(nN_3) $\|\alpha x_1, \ldots, x_n\| = |\alpha| \|x_1, \ldots, x_n\|$ for every $\alpha \in \mathbb{R}$,
(nN_4) $\|x_1 + x_1', x_2 \ldots, x_n\| \leq \|x_1, x_2, \ldots, x_n\| + \|x_1', x_2, \ldots, x_n\|$.
is called an $n-$norm on X. The pair $(X, \|\cdot, \ldots, \cdot\|)$ is called an $n-$normed spaces.

Example 2. *([3]) (i) Let $X = \mathbb{R}^n$ with the following Euclidean $n-$norm:*

$$\|x_1, \ldots, x_n\|_E = abs\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}\right),$$

where $x_i = (x_{i1}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = \overline{1, n}$. Then, the pair $(\mathbb{R}^n, \|x_1, \ldots, x_n\|_E)$ is an $n-$normed space.

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(ii) Let $(X, \|\cdot, \ldots, \cdot\|)$ *be an* $n-$*normed space of dimension* $d \geq n \geq 2$ *and* $\{u_1, u_2, \ldots, u_n\}$ *be a linearly independent set in* X. *A function* $\|\cdot, \ldots, \cdot\|_\infty$ *on* X^{n-1} *defined by*

$$\|x_1, \ldots, x_{n-1}\|_\infty = \max\left\{\|x_1, \ldots, x_n, u_i\| : i = \overline{1, n}\right\}$$

is an $(n-1)$ *norm on* X *w.r.t.* $\{u_1, u_2, \ldots, u_n\}$.

Definition 3. *(*[3]*) Let* X *be a* $n-$*normed linear space and* $\{x_n\}_{n=0}^{\infty}$ *a sequence in* X. *We say that* $\{x_n\}_{n=0}^{\infty}$ *converge to some* $x \in X$ *if*

$$\lim_{n\to\infty} \|x_n - x, u_2, \cdots, u_n\| = 0,$$

for all $u_2, \cdots, u_n \in X$.

The following iterative methods were studied in [6], [7], [8], and [9] respectively,

$$\begin{cases} s_0 \in X, \\ s_{n+1} = a_n s_n + b_n T_1 s_n + c_n T_2 s_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{1}$$

$$\begin{cases} p_0 \in X, \\ p_{n+1} = (1-\alpha_n) q_n + \alpha_n T q_n, \\ q_n = (1-\beta_n) p_n + \beta_n T p_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{2}$$

$$\begin{cases} p_0 \in X, \\ p_{n+1} = (1-\alpha_n) q_n + \alpha_n T_1 q_n, \\ q_n = (1-\beta_n) p_n + \beta_n T_2 p_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{3}$$

$$\begin{cases} x_0 \in X, \\ x_{n+1} = a_n x_n + b_n T_1 y_n + c_n T_2 x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n) T_3 x_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{4}$$

where T, T_1, T_2 and T_3 are self maps of an ambient space X and $\{a_n\}, \{b_n\}, \{c_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in $[0, 1]$ satisfying certain control conditions.

Inspired by the above iterative methods, we introduce the following iterative method.

$$\begin{cases} x_0 \in X, \\ x_{n+1} = a_n y_n + b_n T_1 y_n + c_n T_2 y_n, \\ y_n = \alpha_n x_n + (1-\alpha_n) T_3 x_n, \ \forall n \in \mathbb{N}, \end{cases} . \tag{5}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\} \subset [0, 1]$ are real sequences satisfying $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, and $\sum\limits_{n=0}^{\infty} b_n = \infty$.

Remark 4. *If* $T_3 = I$ *(Identity operator), then iterative method* (5) *reduces to iterative method* (1). *If* $T_1 = I$ *(Identity operator),* $T_2 = T_3 = T$, *then iterative method* (5) *reduces to iterative method* (1.2). *If* $T_1 = I$, *then the iterative method* (5) *reduces to iterative method* (1.3). *Note that* (1.4) *and* (5) *are of independent interest and so we would like to deal with both of these separately. However, it is worth noticing that* (1.4) *does not reduce to* (1.3) *but* (5) *does. Thus, in this sense,* (5) *is more general than* (1.4).

Recently, Dutta [3] introduced a generalized Z-type contractive condition as follows: Let K be nonempty, closed, convex subset of real linear $n-$normed space X and $T : K \to K$ a self map. There exists a constant $L \geq 0$ such that for all $x, y, u_2, \cdots, u_n \in K$, we have

$$\begin{aligned} &\|Tx - Ty, u_2, \cdots, u_n\| \\ &\leq e^{L\|x-Tx,u_2,\cdots,u_n\|} \times (2\delta \|x - Tx, u_2, \cdots, u_n\| + \delta \|x - y, u_2, \cdots, u_n\|), \end{aligned} \tag{6}$$

where $\delta \in [0, 1)$ and e^x denotes the exponential function of $x \in K$.

In [3], some convergence results have been constructed for fixed point of the mappings satisfying condition (6) via iterative schemes (1.1), (1.2) and (1.3).

In this paper, we introduce the following contractive condition: Let $(X, \|\cdot, \ldots, \cdot\|)$ be an $n-$normed space, $T : X \to X$ a selfmap of X, with a fixed point q such that for all $x, y, u_2, \cdots, u_n \in X$ and for some $\delta \in [0, 1)$, we have

$$\|q - Ty, u_2, \cdots, u_n\| \leq \delta \|q - y, u_2, \cdots, u_n\|. \tag{7}$$

This is similar to the condition introduced by [10] and can be obtained from (6) when $x = q$ is a fixed point. We may call this kind of operators quasi-contractive operators.

Following important observation will be used in the sequel.

$$\begin{aligned} \|Tx - Ty, u_2, \cdots, u_n\| &\leq \|Tx - q, u_2, \cdots, u_n\| + \|q - Ty, u_2, \cdots, u_n\| \\ &\leq \delta (\|x - q, u_2, \cdots, u_n\| + \|q - y, u_2, \cdots, u_n\|) \\ &\leq \delta \|x - y, u_2, \cdots, u_n\| + 2\delta \|q - y, u_2, \cdots, u_n\|. \end{aligned} \tag{8}$$

In our opinion, it is better to work with the contractive condition defined by (7) than with (6) because, as remarked above, if we suppose that T has a fixed point, then (6) implies (7) and by using it we can avoid doing unnecessary calculations.

In this paper, we first prove some convergence results for the mappings satisfying condition (7) via iterative methods (1.4) and (5). Next, we show that the iterative methods (1.4) and (5) are stable with respect to (T_1, T_2, T_3). Finally, we prove some data dependence results for the iterative methods (1.4) and (5).

We close this section with the following couple of results useful in proving our main results.

Lemma 5. [11] *Let $\{\sigma_n\}_{n=0}^{\infty}$ and $\{\rho_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the following inequality:*

$$\sigma_{n+1} \leq (1 - \lambda_n) \sigma_n + \rho_n,$$

where $\lambda_n \in (0, 1)$, for all $n \geq n_0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\frac{\rho_n}{\lambda_n} \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} \sigma_n = 0$.

Lemma 6. [12] *Let $\{\sigma_n\}_{n=0}^{\infty}$ be a nonnegative sequence for which one assumes there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$ one has satisfied the inequality*

$$\sigma_{n+1} \leq (1 - \mu_n) \sigma_n + \mu_n \gamma_n,$$

where $\mu_n \in (0,1)$, *for all* $n \in \mathbb{N}$, $\sum\limits_{n=0}^{\infty} \mu_n = \infty$ *and* $\gamma_n \geq 0$, $\forall n \in \mathbb{N}$. *Then the following inequality holds.*

$$0 \leq \limsup_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \gamma_n.$$

2. Convergence Results

For the sake of simplicity, from now on we assume that X is a $n-$normed linear space, T_1, T_2 and T_3 are self-maps of X satisfying the contractive condition (7) with the set of fixed points $F_{T_1}, F_{T_2}, F_{T_3}$ respectively, and $\bigcap\limits_{i=1}^{3} F_{Ti} \neq \emptyset$.

Theorem 7. *Let* $\{x_n\}_{n=0}^{\infty}$ *be a sequence generated by iterative method* (5) *with real sequences* $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\} \subset (0,1)$ *satisfying* $a_n + b_n + c_n = 1$ *for all* $n \in \mathbb{N}$, *and* $\sum\limits_{n=0}^{\infty} b_n = \infty$ (*or* $\sum\limits_{n=0}^{\infty} c_n = \infty$). *Suppose that* $q \in \bigcap\limits_{i=1}^{3} F_{Ti} \neq \emptyset$. *Then the iterative sequence* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to* q.

Proof. First we prove that $q \in \bigcap\limits_{i=1}^{3} F_{Ti}$ is the unique common fixed point of T_1, T_2 and T_3. Suppose that there exists another common fixed point $q^* \in \bigcap\limits_{i=1}^{3} F_{Ti}$. Then from (7), we have

$$\|q - q^*\| = \|T_i q - q^*\| \leq \delta \|q - q^*\| \text{ for each } i = 1, 2, 3,$$

which implies that $q = q^*$ as $\delta \in [0,1)$.

Next, we prove that $x_n \to q$.

Using (5) and (7), we get

$$\begin{aligned} \|x_{n+1} - q, u_2, \cdots, u_n\| &= \|a_n y_n + b_n T_1 y_n + c_n T_2 y_n - q, u_2, \cdots, u_n\| \\ &= \|a_n (y_n - q) + b_n (T_1 y_n - q) + c_n (T_2 y_n - q), u_2, \cdots, u_n\| \\ &\leq a_n \|y_n - q, u_2, \cdots, u_n\| + b_n \|T_1 y_n - q, u_2, \cdots, u_n\| \\ &\quad + c_n \|T_2 y_n - q, u_2, \cdots, u_n\| \\ &\leq [a_n + (b_n + c_n)\delta] \|y_n - q, u_2, \cdots, u_n\| \end{aligned} \tag{9}$$

and

$$\begin{aligned} \|y_n - q, u_2, \cdots, u_n\| &= \|\alpha_n x_n + (1-\alpha_n) T_3 x_n - q, u_2, \cdots, u_n\| \\ &\leq \alpha_n \|x_n - q, u_2, \cdots, u_n\| + (1-\alpha_n) \|T_3 x_n - q, u_2, \cdots, u_n\| \\ &\leq [\alpha_n + (1-\alpha_n)\delta] \|x_n - q, u_2, \cdots, u_n\| \\ &\leq [\alpha_n + 1 - \alpha_n] \|x_n - q, u_2, \cdots, u_n\| \\ &= \|x_n - q, u_2, \cdots, u_n\|. \end{aligned} \tag{10}$$

Substituting (2.2) into (2.1)

$$\|x_{n+1}-q,u_2,\cdots,u_n\| \leq [1-(b_n+c_n)(1-\delta)]\|x_n-q,u_2,\cdots,u_n\|. \tag{11}$$

Since $\delta\in[0,1)$ and $a_n+b_n+c_n=1$ for all $n\in\mathbb{N}$,

$$0\leq 1-(b_n+c_n)(1-\delta)<1. \tag{12}$$

Also, the assumption $\sum\limits_{n=0}^{\infty} b_n=\infty$ (or $\sum\limits_{n=0}^{\infty} c_n=\infty$) implies $\sum\limits_{n=0}^{\infty}(b_n+c_n)=\infty$. Hence, an application of Lemma 1 to (2.3) lead us to $\lim_{n\to\infty} x_n=q$. □

Remark 8. *If $T_1=I$ (Identity operator), $T_2=T_3=T$, then the iterative method* (5) *reduces to the iterative method (1.2). If $T_1=I$, then the iterative method* (5) *reduce to the iterative method (1.3). Having regard to these facts, we conclude that Theorem 1 is a generalization and extension of both (*[3]*, Theorem 3 and Theorem 4).*

Theorem 9. *Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated by iterative method (1.4) with real sequences $\{a_n\},\{b_n\},\{c_n\},\{\alpha_n\}\subset(0,1)$ satisfying $a_n+b_n+c_n=1$ for all $n\in\mathbb{N}$, and $\sum\limits_{n=0}^{\infty} b_n=\infty$ (or $\sum\limits_{n=0}^{\infty} c_n=\infty$). Suppose that $q\in\bigcap\limits_{i=1}^{3}F_{Ti}\neq\emptyset$. Then the iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to q.*

Proof. The proof is quite similar to that of Theorem 1 above, and is thus omitted. □

Remark 10. *If $T_3=I$ (Identity operator), then the iterative method (1.4) reduce to the iterative method (1.1). Thus, we conclude that Theorem 2 is a generalization and extension (*[3]*, Theorem 2).*

3. Stability Results

One of the most studied problems in fixed point theory is the stability of fixed points iterative methods. The initiator of this kind of study seems to be Urabe [13]. Later on, Ostrowski [14] has also put his efforts in this field. However, a formal definition for the stability of general iterative methods was apparently given by Harder and Hicks [15]. Continuing this trend, in the last three decades, a large literature has emerged and developed dealing with the stability of various well-known iterative methods for different classes of operators (see [10, 13-21] and references therein). Below we reformulate the definition of stability given by Harder and Hicks [15] in the context of $n-$normed spaces.

Definition 11. *Let X be a $n-$normed space, T a self map of X, and $\{x_n\}_{n=0}^{\infty}\subset X$ a sequence defined by*

$$x_{n+1}=f(T,x_n),\ n=0,1,\ldots, \tag{13}$$

where $x_0 \in X$ is the initial approximation and f is some function. Suppose that the sequence $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point q of T. Let $\{y_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence and set

$$\varepsilon_n = \|y_{n+1} - f(T, y_n), u_2, \cdots, u_n\|, \; n = 0, 1, \ldots.$$

Then, iteration procedure (3.1) is said to be $T-$stable or stable with respect to T if and only if $\lim_{n\to\infty} \varepsilon_n = 0 \Rightarrow \lim_{n\to\infty} y_n = q$.

Theorem 12. *Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated by the iterative method* (5) *with real sequences $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\} \subset (0,1)$ satisfying $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, and $\sum_{n=0}^{\infty} b_n = \infty$ (or $\sum_{n=0}^{\infty} c_n = \infty$). Suppose that $q \in \bigcap_{i=1}^{3} F_{Ti} \neq \emptyset$. Let $\{r_n\}_{n=0}^{\infty} \subset X$ be any sequence and define a sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ in $\mathbb{R}^+$ by*

$$\begin{cases} \varepsilon_n = \|r_{n+1} - a_n v_n - b_n T_1 v_n - c_n T_2 v_n, u_2, \cdots, u_n\|, \\ v_n = \alpha_n r_n + (1-\alpha_n) T_3 r_n, \; \forall n \in \mathbb{N}. \end{cases} \tag{14}$$

Then, $\{x_n\}_{n=0}^{\infty}$ is stable with respect to (T_1, T_2, T_3).

Proof. Assume that $\lim_{n\to\infty} \varepsilon_n = 0$. In order to prove that $\{x_n\}_{n=0}^{\infty}$ is stable with respect to (T_1, T_2, T_3), it suffices to prove that $\lim_{n\to\infty} r_n = q$.

It follows from (5) and (7) that

$$\begin{aligned} \|r_{n+1} - q, u_2, \cdots, u_n\| &\leq \|r_{n+1} - a_n v_n - b_n T_1 v_n - c_n T_2 v_n, u_2, \cdots, u_n\| \\ &\quad + \|a_n v_n + b_n T_1 v_n + c_n T_2 v_n - q, u_2, \cdots, u_n\| \\ &\leq \varepsilon_n + a_n \|v_n - q, u_2, \cdots, u_n\| \\ &\quad + b_n \|T_1 v_n - q, u_2, \cdots, u_n\| + c_n \|T_2 v_n - q, u_2, \cdots, u_n\| \\ &\leq \varepsilon_n + [a_n + (b_n + c_n)\delta] \|v_n - q, u_2, \cdots, u_n\|, \end{aligned} \tag{15}$$

$$\begin{aligned} \|v_n - q, u_2, \cdots, u_n\| &\leq \alpha_n \|r_n - q, u_2, \cdots, u_n\| + (1-\alpha_n) \|T_3 r_n - q, u_2, \cdots, u_n\| \\ &\leq [\alpha_n + (1-\alpha_n)\delta] \|r_n - q, u_2, \cdots, u_n\|. \end{aligned} \tag{16}$$

Substituting (3.4) in (3.3), we get

$$\|r_{n+1} - q, u_2, \cdots, u_n\| \leq \varepsilon_n + [a_n + (b_n + c_n)\delta][\alpha_n + (1-\alpha_n)\delta] \|r_n - q, u_2, \cdots, u_n\|. \tag{17}$$

Since $\delta \in [0,1)$ and $\alpha_n \in [0,1]$ for all $n \in \mathbb{N}$,

$$\alpha_n + (1-\alpha_n)\delta < 1. \tag{18}$$

Using (3.6) in (3.5), we obtain

$$\begin{aligned} \|r_{n+1} - q, u_2, \cdots, u_n\| &\leq \varepsilon_n + [a_n + (b_n + c_n)\delta] \|r_n - q, u_2, \cdots, u_n\| \\ &= \varepsilon_n + [1 - (b_n + c_n)(1-\delta)] \|r_n - q, u_2, \cdots, u_n\|. \end{aligned}$$

Now using similar arguments as in the proof of Theorem 1, we obtain $\lim_{n\to\infty} r_n = q$ and hence the result. □

Theorem 13. *Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated by the iterative method (1.4) with real sequences $\{a_n\},\{b_n\},\{c_n\},\{\alpha_n\} \subset (0,1)$ satisfying $a_n+b_n+c_n=1$ for all $n \in \mathbb{N}$, and $\sum_{n=0}^{\infty} b_n = \infty$ (or $\sum_{n=0}^{\infty} c_n = \infty$). Suppose that $q \in \bigcap_{i=1}^{3} F_{Ti} \neq \emptyset$. Let $\{r_n\}_{n=0}^{\infty} \subset X$ be any sequence and define a sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ in $\mathbb{R}^+$ by*

$$\begin{cases} \varepsilon_n = \|r_{n+1} - a_n r_n - b_n T_1 v_n - c_n T_2 r_n, u_2, \cdots, u_n\| \\ v_n = \alpha_n r_n + (1-\alpha_n) T_3 r_n, \ \forall n \in \mathbb{N}. \end{cases} \tag{19}$$

Then, $\{x_n\}_{n=0}^{\infty}$ is stable with respect to (T_1, T_2, T_3).

Proof. The proof is quite similar to that of Theorem 3 above, and is thus omitted. □

Remark 14. *If $T_2 = I$ (Identity operator) and $T_1 = T_3 = T$, the iterative method (1.4) reduces to Ishikawa iterative method* [22]*. If $T_1 = T_3 = I$, then the iterative method (1.4) reduce to Mann iterative method* [23]*. Having regard to these facts, we conclude that Theorem 4 is a generalization and extension of both (*[10]*, Theorem 2.2) and (*[16]*, Theorem 2).*

4. Data Dependence Results

In some cases, it is difficult or may even be impossible to find a fixed point of a certain mapping. In such cases, instead of computing the fixed point of this mapping, we approximate this mapping with another one whose fixed point can be easily obtained. Thus we have an estimation for the approximate location of the fixed point of this mapping without actually computing it. For this reason, the topic of data dependency of fixed points has a great importance both from numerical and theoretical perspectives. Consequently, the study of data dependence of fixed points in a normed space setting has attracted several researchers; (see [12, 17, 25] and references therein).

Definition 15. *Let X be a $n-$normed space, $T, \widetilde{T} : X \to X$ two operators. We say that $\widetilde{T}$ is an approximate operator of T if for all $x, u_2, \cdots, u_n \in X$ and for a fixed $\varepsilon > 0$, we have*

$$\left\| Tx - \widetilde{T}x, u_2, \cdots, u_n \right\| \leq \varepsilon.$$

Theorem 16. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by the iterative method* (5) *associated to T_1, T_2 and T_3 with a common fixed point $q \in \bigcap_{i=1}^{3} F_{Ti} \neq \emptyset$, and $\{\widetilde{x}_n\}_{n=0}^{\infty}$ be the iterative sequence generated by*

$$\begin{cases} \widetilde{x}_0 \in X, \\ \widetilde{x}_{n+1} = a_n \widetilde{y}_n + b_n \widetilde{T}_1 \widetilde{y}_n + c_n \widetilde{T}_2 \widetilde{y}_n \\ \widetilde{y}_n = \alpha_n \widetilde{x}_n + (1-\alpha_n) \widetilde{T}_3 \widetilde{x}_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{20}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\} \subset [0,1]$ *are real sequences satisfying* $a_n + b_n + c_n = 1$ *and* $\frac{1}{2-\delta} \leq b_n$ *(or* $\frac{1}{2-\delta} \leq c_n$ *) for all* $n \in \mathbb{N}$. *Suppose that for fixed* $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ *and for all* $x, u_2, \cdots, u_n \in X$, *we have* $\left\| T_1 x - \widetilde{T}_1 x, u_2, \cdots, u_n \right\| \leq \varepsilon_1$, $\left\| T_2 x - \widetilde{T}_2 x, u_2, \cdots, u_n \right\| \leq \varepsilon_2$, $\left\| T_3 x - \widetilde{T}_3 x, u_2, \cdots, u_n \right\| \leq \varepsilon_3$.

If $q^* \in \bigcap_{i=1}^{3} F_{\widetilde{T}i} \neq \emptyset$ *such that* $\widetilde{x}_n \to q^*$ *as* $n \to \infty$, *then we have*

$$\| q - q^*, u_2, \cdots, u_n \| \leq \frac{3\varepsilon}{1-\delta},$$

where $\varepsilon = \max\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$.

Proof. It follows from (5), (4.1), (7), and (8) that

$$\begin{aligned} \| x_{n+1} - \widetilde{x}_n, u_2, \cdots, u_n \| &\leq a_n \| y_n - \widetilde{y}_n, u_2, \cdots, u_n \| \\ &\quad + b_n \left\| T_1 y_n - \widetilde{T}_1 \widetilde{y}_n, u_2, \cdots, u_n \right\| \\ &\quad + c_n \left\| T_2 y_n - \widetilde{T}_2 \widetilde{y}_n, u_2, \cdots, u_n \right\|, \end{aligned} \tag{21}$$

$$\begin{aligned} \left\| T_1 y_n - \widetilde{T}_1 \widetilde{y}_n, u_2, \cdots, u_n \right\| &\leq \| T_1 y_n - T_1 \widetilde{y}_n, u_2, \cdots, u_n \| \\ &\quad + \left\| T_1 \widetilde{y}_n - \widetilde{T}_1 \widetilde{y}_n, u_2, \cdots, u_n \right\| \\ &\leq 2\delta \| y_n - q, u_2, \cdots, u_n \| \\ &\quad + \delta \| y_n - \widetilde{y}_n, u_2, \cdots, u_n \| + \varepsilon_1, \end{aligned} \tag{22}$$

$$\begin{aligned} \left\| T_2 y_n - \widetilde{T}_2 \widetilde{y}_n, u_2, \cdots, u_n \right\| &\leq 2\delta \| y_n - q, u_2, \cdots, u_n \| \\ &\quad + \delta \| y_n - \widetilde{y}_n, z_2, \ldots, z_n \| + \varepsilon_2, \end{aligned} \tag{23}$$

$$\| y_n - q, u_2, \cdots, u_n \| \leq [\alpha_n + (1-\alpha_n)\delta] \| x_n - q, u_2, \cdots, u_n \|, \tag{24}$$

$$\begin{aligned} \| y_n - \widetilde{y}_n, u_2, \cdots, u_n \| &\leq \alpha_n \| x_n - \widetilde{x}_n, u_2, \cdots, u_n \| \\ &\quad + (1-\alpha_n) \left\| T_3 x_n - \widetilde{T}_3 \widetilde{x}_n, u_2, \cdots, u_n \right\|, \end{aligned}$$

$$\begin{aligned} \left\| T_3 x_n - \widetilde{T}_3 \widetilde{x}_n, u_2, \cdots, u_n \right\| &\leq 2\delta \| x_n - q, u_2, \cdots, u_n \| \\ &\quad + \delta \| x_n - \widetilde{x}_n, u_2, \cdots, u_n \| + \varepsilon_3. \end{aligned} \tag{25}$$

Combining (4.2)-(4.6)

$$\begin{aligned} \| x_{n+1} - \widetilde{x}_n, u_2, \cdots, u_n \| &\leq [a_n + (b_n + c_n)\delta] [\alpha_n + \delta(1-\alpha_n)] \| x_n - \widetilde{x}_n, u_2, \cdots, u_n \| \\ &\quad + \{ (b_n + c_n) [\alpha_n + (1-\alpha_n)\delta] \\ &\quad + (1-\alpha_n) [a_n + (b_n + c_n)\delta] \} 2\delta \| x_n - q, u_2, \cdots, u_n \| \\ &\quad + b_n \varepsilon_1 + c_n \varepsilon_2 + (1-\alpha_n) [a_n + (b_n + c_n)\delta] \varepsilon_3, \end{aligned} \tag{26}$$

Since $\delta \in [0,1)$ and $\alpha_n, a_n, b_n, c_n \in [0,1]$ for all $n \in \mathbb{N}$,we have

$$\begin{cases} 1-\alpha_n \leq 1, \\ b_n \leq b_n + c_n, \\ c_n \leq b_n + c_n. \end{cases} \tag{27}$$

Using (2.4), (3.6), (4.8) and assumption $\frac{1}{2-\delta} \leq b_n$ (which implies $\frac{1}{2-\delta} \leq b_n + c_n$) for all $n \in \mathbb{N}$, (4.7) becomes

$$\begin{aligned} \|x_{n+1} - \widetilde{x}_n, u_2, \cdots, u_n\| &\leq [1 - (b_n + c_n)(1-\delta)] \|x_n - \widetilde{x}_n, u_2, \cdots, u_n\| \\ &\quad + (b_n + c_n)(1-\delta) \\ &\quad \times \frac{\{(2-\alpha_n) 2\delta \|x_n - q, u_2, \cdots, u_n\| + \varepsilon_1 + \varepsilon_2 + \varepsilon_3\}}{1-\delta}. \end{aligned} \tag{28}$$

Define

$$\begin{aligned} \sigma_n &= \|x_n - \widetilde{x}_n, u_2, \cdots, u_n\|, \\ \mu_n &= (b_n + c_n)(1-\delta) \in (0,1), \\ \gamma_n &= \frac{(2-\alpha_n) 2\delta \|x_n - q, u_2, \cdots, u_n\| + \varepsilon_1 + \varepsilon_2 + \varepsilon_3}{1-\delta}, \text{ for all } n \in \mathbb{N}. \end{aligned}$$

As the assumption $\frac{1}{2-\delta} \leq b_n$ (or $\frac{1}{2-\delta} \leq c_n$) implies $\sum_{n=0}^{\infty} b_n = \infty$ (or $\sum_{n=0}^{\infty} c_n = \infty$), we have $\sum_{n=0}^{\infty} (b_n + c_n) = \infty$ as in the proof of Theorem 1. Thus all conditions in Lemma 2 are satisfied by (4.9). Also, from Theorem 1, we know that $x_n \to q$ as $n \to \infty$. Hence, we have

$$\|q - q^*, u_2, \cdots, u_n\| \leq \frac{3\varepsilon}{1-\delta},$$

where $\varepsilon = \max\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. □

Theorem 17. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by the iterative method (1.4) associated to T_1, T_2 and T_3 with a common fixed point $q \in \bigcap_{i=1}^{3} F_{Ti} \neq \emptyset$, and $\{\widetilde{x}_n\}_{n=0}^{\infty}$ be the iterative sequence generated by*

$$\begin{cases} \widetilde{x}_0 \in X, \\ \widetilde{x}_{n+1} = a_n \widetilde{x}_n + b_n \widetilde{T}_1 \widetilde{y}_n + c_n \widetilde{T}_2 \widetilde{x}_n, \\ \widetilde{y}_n = \alpha_n \widetilde{x}_n + (1-\alpha_n) \widetilde{T}_3 \widetilde{x}_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{29}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\} \subset [0,1]$ are real sequences satisfying $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, and $\sum_{n=0}^{\infty} b_n = \infty$ (or $\sum_{n=0}^{\infty} c_n = \infty$). Suppose that for fixed $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ and for all $x, u_2, \cdots, u_n \in X$, we have

$$\left\| T_1 x - \widetilde{T}_1 x, u_2, \cdots, u_n \right\| \leq \varepsilon_1, \left\| T_2 x - \widetilde{T}_2 x, u_2, \cdots, u_n \right\| \leq \varepsilon_2,$$

$$\left\| T_3x - \widetilde{T}_3x, u_2, \cdots, u_n \right\| \quad \leq \quad \varepsilon_3.$$

If $q^* \in \bigcap_{i=1}^{3} F_{\widetilde{T}i} \neq \emptyset$ *such that* $\widetilde{x}_n \to q^*$ *as* $n \to \infty$*, then we have*

$$\|q - q^*, u_2, \cdots, u_n\| \leq \frac{(2+\delta)\,\varepsilon}{1-\delta},$$

where $\varepsilon = \max\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$.

Proof. The proof is quite similar to that of Theorem 5 above, and is thus omitted. □

Remark 18. *If* $a_n = 0$ *and* $T_1 = T_2 = T_3 = T$*, then iterative method (1.3) reduce to* $S-$*iterative method* [24]*. Also, keeping in mind Remark 3, Theorem 6 generalizes both (*[12]*, Theorem 3.2) and (*[25]*, Theorem 4).*

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